Input third-degree price discrimination by congestible facilities

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Abstract

This paper studies third-degree price discrimination by transport facilities, such as airports and seaports, which sell access to the infrastructure as a necessary input for downstream production. These facilities are prone to congestion—which makes downstream markets interrelated—and their ownership structure is diverse, varying from public (domestic welfare maximizing) to private (profit maximizing). We show that input price discrimination by a private supplier can increase aggregate output and increase welfare in a setting where, in absence of congestion, output does not change and welfare is reduced when price discrimination is allowed. Therefore, the presence of negative consumption externalities enlarges the extent to which input price discrimination is desirable. We also analyze the effects of price discrimination by a public facility and describe the conditions under which banning input price discrimination is efficient for both types of ownership forms. We argue that there is a limited scope for this to occur, which suggests that the current practice of enforcing a broad ban on input price discrimination that covers congestible facilities with different ownership forms may have to be revised.

1. Introduction

Congestible facilities often provide infrastructure access to downstream firms that may operate in different markets. For example, access to the airport’s runway is an essential input for an airline’s production in multiple city-pairs. In many countries, input price discrimination is banned by law, so that suppliers must charge uniform prices to firms. To a large extent the ban on input price discrimination applies also to congestible facilities. For example, the EU Airport Charges directive (2009/12/EC) prohibits differentiated charges to airlines using the same service (i.e. terminal and level of service).¹ A similar ban holds for airports in the U.K. (Section 41 of the 1986 Airports Act) and in the U.S. (2013 FAA’s Policy

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¹The ban applies to the airport with the highest passenger movement in each EU Member State and to any airport whose annual traffic is over 5 million passengers.
Regarding Airport Rates and Charges).² The regulations of the World Trade Organization (WTO) through the General Agreement on Tariffs and Trade (GATT) basically do not allow price discrimination by ports. Similar examples can be found in other transport sectors too. It is therefore evident that current bans on price discrimination to congestible facilities have an impact on many large economic sectors.

Congestible facilities feature two characteristics that make the analysis distinct from the traditional price discrimination studies in input markets. First, there is congestion: an output increase by one firm imposes additional costs on consumers of all markets, therefore reducing the price that other firms can charge. Congestion, thus, makes demands inter-related in a way analogous to substitution. In addition, downstream firms do not fully internalize this externality, so that the aggregate output may be inefficiently high. Second, the ownership form of the congestible facilities subject to price discrimination regulation is diverse: for example, in Europe alone, the ban applies to private, public and mixed private-public airports. The incentives of the facilities to apply price discrimination and, therefore, its effect on welfare may vary with the ownership form. The purpose of this paper is to study third-degree price discrimination by both private (profit maximizing) and public (domestic welfare maximizing) congestible facilities, and shed light on whether and when a broad (e.g. Europe-wide) ban on input price discrimination is desirable.

In order to focus on the effects of congestion and ownership form, we analyze a case with two downstream markets and two downstream firms, one domestic and one foreign, that are equally efficient. Each firm is a monopoly in one market and the only interdependency is through congestion. To highlight the differences with the uncongested case, we consider an industry structure that is comparable to the commonly used structure in the literature in that the input provider is a monopolist and firms take the input price as given. A private facility would therefore differentiate charges according to the different demand conditions in the two markets. A public facility also considers domestic consumer surplus and the profit of the domestic firm, so it would give price concessions to the domestic firm in detriment of the foreign firm, to stimulate (domestic) production and capture foreign profit.

In this paper we, first, analyze the price, output and welfare effect of input third-degree price discrimination by a private facility and assess how the presence of congestion externalities affects the analysis. Second, we study the case of a public facility and assess the impact of the ownership form on the effects of price discrimination. Finally, we compare the welfare effect of price discrimination under both ownership forms and elaborate on the desirability of a broad ban on price discrimination.

Input third-degree price discrimination when downstream firms are equally efficient and operate in multiple markets has been recently explored by Arya and Mittendorf (2010). In

²The FAA’s Policy Regarding Airport Rates and Charges prohibits “unjust discrimination”. This prohibition does not prevent airports to set different charges to different aeronautical users (such as signatory and nonsignatory carriers) or to in peak and off-peak periods. Nevertheless, it explicitly bans, for example, differentiated charges to firms that belong to the same category irrespective of the markets they serve, and to foreign and domestic airlines engaged in similar international air services.
a two-market setting and using linear demands, they show that the aggregate output is the
same under both pricing regimes (i.e. with and without discrimination), but price discrim-
ination leads to an output shift from the market with higher demand to the market with
lower demand. Therefore, extending the intuition from final good markets, price discrim-
ination leads to welfare deterioration in the case where there is only one firm operating
in each market.\textsuperscript{3,4} Our paper contributes to this branch of the literature by studying
the case with interrelated demands through congestion and by considering a domestic-welfare-
maximizing input provider. Also with linear demands, yet in the presence of congestion,
we find benefits from price discrimination when the willingness to pay to reduce travel
delays differs across markets. This may be due to different average income of consumers
or different composition of trip purpose (leisure versus business) across markets, among
others. We show that under private and public ownership, input price discrimination can
increase aggregate output and welfare. This result suggests that the presence of congestion
externalities enlarges the extent to which input price discrimination is desirable.

The literature on price discrimination under negative consumption externalities has
mainly focused on final markets. Adachi (2005), considering only consumption externali-
ties within markets, shows that welfare can increase when third-degree price discrimination
is allowed when output does not. Czerny and Zhang (2015) study third-degree price discrim-
ination by a monopoly airline considering cross- and within-market negative externalities
together, a feature that is typical of congestion. They show that there is a time-valuation
effect of price discrimination that works in the opposite direction as the output effect and,
as a result, welfare can increase when output decreases. When demands are linear, they find
that price discrimination reduces the aggregate passenger quantity, which reduces conges-
tion costs, and that this can increase welfare. We show that price discrimination in input
markets under congestion externalities exhibits fundamental differences with the case of
final markets and that the analysis provides essentially different insights. We find that un-
der linear demands, input third-degree price discrimination by a profit maximizing facility
can yield higher total welfare and consumer surplus than what is obtained under uniform
pricing by leading to aggregate output expansion and to a reduction of both prices. As
congestion effects work in a similar way as the substitution effect, the intuition is similar
to the one provided by Layson (1998) for substitute final goods. Reducing the price in one
market can reduce the profitability of the other and, if this effect is large enough, it can
cause that the price in the other market has to also be reduced. These results are in sharp

\textsuperscript{3}They also analyze the case in which a firm operates in both markets and faces different degrees of
competition in each one. When the market with lower demand is also the market with lower competition,
the increased production incentives under price discrimination in this market may increase welfare.

\textsuperscript{4}There is also a large stream of literature studying third-degree price discrimination in input markets
where downstream firms have different levels of (cost) efficiency that shows benefits of uniform pricing (e.g.,
Katz, 1987; DeGraba, 1990; Yoshida, 2000; Valletti, 2003). Nevertheless, uniform pricing can be harmful
when there is bargaining between buyers and suppliers (O’Brien and Shaffer, 1994), and when there is input
demand-side substitution (Inderst and Valletti, 2009).
contrast with the outcome of the models in final markets. The difference arises because the input provider faces derived demands, which may have essential differences with final good demands. For example, under price discrimination by a private supplier, the market with the highest input price can be the market with the lowest final good price. Therefore, what could be called the “weak” market in terms of final price can be the “strong” market for the input provider.

Our analysis also contributes to the transport policy literature. Benoot et al. (2013) study price discrimination by a local welfare maximizing airport when passengers are homogenous and airlines and markets are symmetric. The incentives for price discrimination arise from that the foreign passengers’ surplus is not fully considered. They numerically find that welfare is higher under uniform pricing because foreign passengers surplus increases. We focus on the more fundamental question of whether and when a broad ban on price discrimination is welfare enhancing. We find that under uniform pricing total welfare may be higher than under price discrimination by a domestic welfare maximizing facility also when there is asymmetry, but this is not the only possible outcome as price discrimination can increase welfare. Importantly, we find that under the conditions that make uniform pricing by a public facility welfare enhancing, price discrimination may yield higher total welfare than uniform pricing if the input supplier is private. We also find that the reverse may happen: price discrimination by a public facility can increase total welfare under the same conditions that make uniform pricing socially optimal if the supplier is private. These results have important policy implications. The ownership form of transport facilities has been consistently moving from public to private in the last decades; for example, in 2010, 48% of all European traffic was handled by a fully privatized airport or by mixed private-public airports. Our results suggest that a ban on price discrimination that covers a large number of congestible facilities and, in particular, that covers different ownership forms has to be revised, especially in the light of the privatization wave.

The remainder of the paper is structured as follows. Section 2 introduces the model and main assumptions. Section 3 analyzes the effects of price discrimination by a private facility while Section 4 analyzes the case of a public facility. Section 5 compares the welfare effect of price discrimination under both ownership forms. Section 6 extends the conclusions to downstream perfect price discrimination and Section 7 concludes.

2. The model and the downstream markets

We study price discrimination by a monopolist transport facility that sells access to its infrastructure, which is an input necessary for downstream production. There are two downstream markets served by the facility, $A$ and $B$, which may represent movement of
people or cargo to different destinations. Markets are interrelated through congestion as an additional unit of output in any market imposes an externality on all other consumers, but are otherwise independent. Downstream firms transform one unit of input into one unit of output. Think, for example, of an airport setting per-passenger charges to airlines flying to different cities, where congestion occurs at the passenger’s facilities (security passenger and baggage screening or access to gateways) and/or on the runway as a result of aircraft landing and take-off.

There are two downstream firms and we denote them in the same way as the markets in which they operate. Thus, firm $i$ operates in market $i$ with $i = \{A, B\}$. Each firm’s demand $q_i$ depends on the full price faced by consumers, which is the sum of the downstream firm’s price (e.g. ticket) and the cost of congestion (e.g. delays at the airport). The delay due to congestion, $D(Q)$, increases in the aggregate consumption ($Q = q_A + q_B$), to reflect within- and cross-market negative consumption externalities. As every unit of output causes delays on all others, a natural interpretation for the market demand is that it is the aggregation of consumers who buy either 0 or 1 unit of the good (e.g. a trip in a peak period) and are heterogeneous in their willingness to pay for the good. Denote $P_i(q_i)$ the downward-sloping inverse demand in market $i$ and $v_i$ the willingness to pay to reduce congestion delays, or the value of time as shorthand, which is assumed to be the same for all individuals in a market but to be different across markets. Without loss of generality we assume that consumers in market $A$ have a higher time valuation than consumers in market $B$, so that $v_A > v_B$ holds in the remainder of the paper. We also assume that downstream firms have constant marginal costs and, following Singh and Vives (1984), that their costs are incorporated through the intercept of the inverse demand function.

In the analysis that follows, we study the case where downstream firms set a unit price and cannot discriminate consumers so, in equilibrium, the firm’s price equals $P_i(q_i) - v_i \cdot D(Q)$, the marginal willingness to pay net of congestion delay costs. Section 6 extends the analysis by relaxing this assumption and studies downstream firms applying first-degree price discrimination. Consequently, for a given input price, $w_i$, the downstream firm $i$ maximizes:

$$\pi_i = q_i \cdot [P_i(q_i) - v_i \cdot D(Q) - w_i],$$

and the first-order condition leads to the following pricing rule:

$$\frac{\partial \pi_i}{\partial q_i} = 0 \Rightarrow P_i(q_i) - v_i \cdot D(Q) = w_i + q_i \cdot \left[-P_i'(q_i) + v_i \cdot D'(Q)\right].$$

Eq. (2) shows that the firm’s pricing rule has the facility charge ($w_i$), a traditional monopoly market power markup ($-q_i \cdot P_i'$) and the marginal congestion cost on firm $i$’s own consumers.

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6 We assume that congestion does not affect the downstream firms’ costs, but this could be readily included in our analysis without changing the main results and conclusions. The reason is that congestion does affect firms in that increased congestion raises the full price faced by consumers and therefore final good prices will be lowered by the increased congestion. In the downstream firms’ profit function, whether congestion raises the costs or reduces the passengers’ willingness to pay makes no difference.

7 If $a_i$ is the inverse demand intercept in market $i$ and $c_i$ the marginal cost, we may replace $A_i$ by $a_i - c_i$. 5
(\(q_i \cdot v_i \cdot D'\)). The downstream firm realizes that an additional consumer raises congestion and reduces the price it can charge, but does not internalize the effect on the other firm’s consumers. This internalization result was first recognized by Daniel (1995) in the context of airport congestion pricing and explored theoretically by Brueckner (2002). The system of first-order conditions in Eq. (2) for both firms defines the derived demands faced by the input provider \(q_A(w_A, w_B)\) and \(q_B(w_A, w_B)\). The closed form for the derived demands are in Appendix A.

Before moving into the supplier’s maximization problem, it is useful to compare the downstream price with the welfare maximizing price. In this model, total welfare is:

\[
W = \sum_i \left[ \int_0^{q_i} P_i(x)dx \right] - \left[ \sum_i v_i \cdot q_i \right] \cdot D(Q),
\]

and the welfare maximizing downstream pricing rule is

\[
\frac{\partial W}{\partial q_i} = 0 \Rightarrow P_i(q_i) - v_i \cdot D(Q) = [q_A \cdot v_A + q_B \cdot v_B] \cdot D'(Q) \ \forall \ i \in \{A, B\}. \tag{4}
\]

A comparison between Eq. (2) and Eq. (4) reveals that, input prices aside, the prices set by downstream firms are not necessarily higher than optimal. If the demand is sufficiently elastic, i.e. the demand-related markup is low compared to the un-internalized externality (e.g. \(-q_A \cdot P'_A < q_B \cdot v_B \cdot D'\)), prices will be too low and output too high. This result and its implications for airport pricing have also been discussed in the air transportation literature (see e.g. Pels and Verhoef, 2004).

As one of our aims is to analyze the role of congestion on the effects of input price discrimination, we follow much of the literature and assume that demands are linear. This allows us to compare our results to those in the previous literature more transparently, as mainly models with linear demands have been used to study the effect of price discrimination in final goods markets under negative consumption externalities (e.g. Adachi, 2005) and when input buyers participate in multiple markets (e.g. Arya and Mittendorf, 2010). We also assume, for simplicity, that the congestion delay function is linear in the aggregate quantity. Note that assuming linear functional forms does not mean that we confine our analysis to a constant aggregate output, because in presence of within- and cross-market congestion externalities and linear demands, the output effect of price discrimination is not zero when time valuations are different (Czerny and Zhang, 2015).

The pricing regimes that we study are uniform pricing, where the facility is restricted to charge all firms the same price per unit of output, and price discrimination, where the facility is allowed to charge different unit prices.\(^8\) We assume throughout the paper that all markets are always served under both pricing regimes. The equilibrium concept that we use is subgame-perfect Nash equilibrium, and we use backward induction to identify it. We first study the case of a profit maximizing facility.

\(^8\)There is a distinction between price differentiation and price discrimination in congestible markets (see e.g. van der Weijde, 2014). As in our setting the marginal external cost (\(\sum_i v_i \cdot q_i \cdot D'(Q)\)) is the same for all consumers, there is no difference between discrimination and differentiation, so we use them interchangeably.
3. Private facility

3.1. Price discrimination

When price discrimination is allowed, the facility chooses $w_A$ and $w_B$ to maximize:

$$\Pi^{PD} = w_A \cdot q_A(w_A, w_B) + w_B \cdot q_B(w_A, w_B).$$

where we normalize the input supplier’s costs to zero. The first-order conditions lead to the closed-form solutions for $w_A$ and $w_B$ (see Appendix A) and imply the following pricing rules:

$$w_A = 2 \cdot q_A \left[ -P_A' + v_A \cdot D' \right] + q_B \cdot v_B \cdot D',
$$

$$w_B = 2 \cdot q_B \left[ -P_B' + v_B \cdot D' \right] + q_A \cdot v_A \cdot D'.
$$

Not surprisingly, the input provider also exerts market power and consumers face a double marginalization. In addition, the facility charges the marginal congestion cost that is not internalized by the firm (the last term on the right-hand side of Eqs. (6) and (7)). Therefore, under price discrimination, the final price in each market is higher than the socially optimal price and output is inefficiently low. This result is useful for the welfare analysis below and it is essentially different to the case of final good markets and congestion externalities, where the quantity under price discrimination can be inefficiently high. This is because the downstream firm’s markup is not necessarily higher than the marginal external congestion cost, but the sum of the downstream and upstream markups is.

In our analysis a crucial aspect is whether $w_B$, the input price in the market with a lower value of time, is higher than $w_A$, the price in the market with higher value of time. In absence of congestion effects and, therefore, of interrelation between markets, the input price is higher in the market with the higher inverse demand intercept (see e.g. Arya and Mittendorf, 2010). This is because with linear demands, the intercept determines the elasticity of the derived demand faced by the input supplier. The market with the higher inverse demand intercept is the less elastic market under uniform prices and, therefore, the market where the discriminating input price will be higher (the “strong” market). We seek to understand what is the effect of the congestion externality on this. Let $A_i$ be the intercept of the inverse demand function for market $i$. Assuming that the second-order conditions are satisfied, the following lemma summarizes the condition for $w_B > w_A$ to hold (the proof of this Lemma and all other proofs required in this section are in Appendix A):

**Lemma 1.** The input price under price discrimination is higher in the market with a lower value of time ($w_B > w_A$ holds) if, and only if,

$$\frac{A_B}{A_A} > \lambda_1 = \frac{8 \cdot P_A' \cdot P_B' + 5 \cdot v_A \cdot v_B \cdot D'^2 + v_A^2 \cdot D'^2 + 2 \cdot v_B \cdot D' \cdot v_A \cdot D' \cdot [4 \cdot P_A' - P_B'] - 6 \cdot v_A \cdot v_B \cdot D'}{8 \cdot P_A' \cdot P_B' + 5 \cdot v_A \cdot v_B \cdot D'^2 + v_A^2 \cdot D'^2 + 2 \cdot v_B \cdot D' \cdot v_A \cdot D' \cdot [4 \cdot P_B' - P_A'] - 6 \cdot v_A \cdot v_B \cdot D'},$$

with $\lambda_1 < 1$.

9A sufficient condition is that time valuations are not too distinct in that $v_B/v_A > 7 - 4\sqrt{3} \approx 0.072.$
To understand the intuition behind the Lemma, first consider the case where time valuations are the same in both markets \((v_A = v_B)\). In this case, \(\lambda_1 = 1\) holds and the result obtained in absence of interrelation goes through. When cross congestion effects are symmetric the supplier’s incentive to charge a higher price in one market over the other do not change and the input price is higher in the market where the reservation price is higher. Second, consider the case where the reservation price is the same in both markets \((A_B/A_A = 1)\), a case where, in absence of congestion and interrelation, it is optimal for the supplier to set a uniform price because the elasticities of the derived demand are the same. If there were only within-market congestion externalities (i.e. absence of interrelation), as in Adachi (2005), it would also be optimal for the facility to set a uniform price. Adachi (2005) shows in final good markets that the price is higher in the market with higher reservation price because it fully determines which market is less elastic when consumption externalities are linear in the quantity. In the case of input markets, this is also the case as it is straightforward to show that the differences in elasticity of the input demand can be fully explained by differences in the reservation price due to the linear demand and congestion assumption. Thus, cross congestion effects drive the incentive to set a higher price in one market over the other when \(A_B/A_A = 1\). In our setting, raising the price in one market causes a decrease in congestion costs through decreased demand, which, in turn, causes an increase in the profitability of the other market as the willingness to pay is increased. Consequently, when the reservation price is the same in both markets, it is optimal for the input supplier to set a higher price in the market with low time valuation. Phrased differently, for the input supplier the decreased congestion is more profitable in the market with high time valuations because the increase in willingness to pay is higher. Third, consider the case of different inverse demand intercepts and time valuations, where both effects come into play as Lemma 1 reveals. A lower demand intercept makes the input demand more elastic as it is normally the case with linear demands (in absence of congestion), which gives incentives to decrease the input price, and a lower value of time gives incentives to increase the price in that market because of cross congestion effects. It is straightforward to show that \(\lambda_1\) decreases as the ratio \(v_B/v_A\) is lower, so that the more asymmetric the congestion effects are, the stronger the incentives to raise the price in market \(B\). This is why \(\lambda_1 < 1\) and even when the reservation price is larger in market \(A\), the input price can be higher in market \(B\).

In the general case, the interplay between the relative size of the inverse demand slopes, the time valuations and the inverse demand slopes determines which market faces a higher input price. Interestingly, it is also straightforward to show that \(\lambda_1 > v_B/v_A\) holds regardless of the relative size of the inverse demand slopes. Therefore, Lemma 1 implies that it is more likely that the input price is higher in market \(B\) if the asymmetry of inverse demand intercepts is lower than the asymmetry of time valuations. If they are similar or \(A_B/A_A\) is less than the ratio of time valuations, then the input price will be higher in market \(A\). For the congestion effects to overturn the incentives provided by different demand intercepts (elasticities in absence of congestion), the difference between time valuations must be higher.
than the difference between demand intercepts.

3.2. Uniform pricing

Under a uniform pricing regime, the profit-maximizing facility maximizes:

$$\Pi^U = w \cdot [q_A(w, w) + q_B(w, w)] ,$$

and the first-order condition leads to the following pricing rule (again prices are not informative and are in Appendix A):

$$w = \frac{2 \cdot q_A \cdot \left[ -P_A' + v_A \cdot D' \right] \cdot -P_B' + v_B \cdot D' - P_A' + v_A \cdot D'}{-P_A' - P_B' + \bar{v} \cdot D'} + \frac{2 \cdot q_B \cdot \left[ -P_B' + v_B \cdot D' \right] \cdot -P_A' + v_A \cdot D'}{-P_A' - P_B' + \bar{v} \cdot D'} \frac{(q_A + q_B)}{2} \cdot \frac{[v_A \cdot D'] \cdot [v_B \cdot D']}{-P_A' - P_B' + \bar{v} \cdot D'} ,$$

(9)

where $\bar{v} = (v_A + v_B)/2$ is the average value of time.

The pricing rule in Eq. (9) includes a weighted sum of the supplier’s markups and a negative term that is also a weighted sum of the marginal congestion cost that is external to each firm. It is straightforward to show that the uniform price in Eq. (9) is not an weighted average of the differentiated prices in Eqs. (6) and (7). What causes this result is the demand interdependency through congestion. As Czerny and Zhang (2015) show in final good markets, the uniform price can be lower than both discriminatory prices in presence of congestion externalities. We obtain a similar result in our setting of input price discrimination, so the uniform price is not necessarily an average of the differentiated prices because of demand interrelation. In the following section we study the relationship between uniform and discriminatory prices in detail.

3.3. The effects of price discrimination on prices and output

To study the effect of price discrimination on input prices and output, we use the price-difference constraint method used by Leontief (1940) and Schmalensee (1981). We assume that the facility maximizes profit subject to the constraint $w_B - w_A \leq t$. This is, the input supplier cannot differentiate prices more than the exogenous amount $t \geq 0$. When $t = 0$, the facility sets the uniform price derived above (Eq. (9)). As $t$ gradually increases, the input supplier is gradually allowed to increase the price differentiation until it reaches a point, $t^*$, where it sets the prices $w_A$ and $w_B$ in Eqs. (6) and (7). The method consists of evaluating the marginal effect of relaxing the constraint on a variable, such as aggregate output. If the sign of the marginal effect does not change in the range $[0, t^*]$, the overall effect of price discrimination on the variable will have the same sign, as long as the unrestricted input provider sets a higher charge in market $B$ ($w_B > w_A$). If the opposite holds, i.e. $w_B < w_A$, the overall effect of price discrimination will have the opposite sign of the marginal effect, because the price discrimination behavior is approached by making $t$ negative. All the derivations needed for the results in this section are in Appendix A.
For a given value of \( t \in [0, t^*] \), the facility maximizes:

\[
\Pi = w_A \cdot q_A(w_A, w_A + t) + (w_A + t) \cdot q_B(w_A, w_A + t) .
\]

(10)

Totally differentiating the first-order condition \( \partial \Pi / \partial w_A \), we can obtain the marginal effect on the aggregate output and input prices:

\[
\begin{align*}
\frac{dQ}{dt} &= \frac{[v_A - v_B] \cdot D'}{2 \cdot \Omega_1} > 0, \\
\frac{dw_A}{dt} &= \frac{4 \cdot P'_A - [3 \cdot v_A - v_B] \cdot D'}{\Omega_2} < 0, \\
\frac{dw_B}{dt} &= \frac{[3 \cdot v_B - v_A] \cdot D' - 4 \cdot P'_B}{\Omega_2},
\end{align*}
\]

(11) (12) (13)

where \( \Omega_1 \) and \( \Omega_2 \) are positive constants. The results that follow from Eqs. (11)–(13) are summarized in the following proposition.

**Proposition 1.** When demands are such that the facility sets a higher input price in the market whose consumers have a lower value of time (i.e. \( w_B > w_A \)), price discrimination:

(i) Increases aggregate output.
(ii) Decreases the input price in the market where time valuations are higher (A).
(iii) Decreases both input prices if time valuations are sufficiently different in that \( v_A - 3 \cdot v_B > -4 \cdot P'_B / D' \); otherwise, it increases the input price in the market where time valuations are lower (B).

When demands are such that \( w_A > w_B \) holds, the effects are reversed and price discrimination decreases output, increases the input price in market A, and it increases both prices when time valuation are sufficiently different.

Our output effect result is an extension of the result in Layson (1998), who shows, in substitute final good markets, that under linear demands the sign of the output effect is determined by the relative magnitude of the gross substitution effect. In our setting outputs are not substitutes nor complements, but the interdependency through congestion generates a similar effect as substitution. An output increase in one market increases the full price of the other market’s consumers by means of increased congestion and therefore it induces an output reduction. As the cross effects are proportional to the time valuations (see Appendix A), it is intuitive that the output change is not zero as long as these effects are not symmetric (\( v_A \neq v_B \)). A difference with Layson (1998) is that the relative magnitude of the cross effects is not the only determinant of the sign of the output effect. That is, if \( v_A > v_B \) holds and the relative magnitude of the cross effects is given, output may rise or fall depending on the relative magnitude of the input prices (the sign of \( w_B - w_A \)).

This result on the output effect also extends previous analyses of price discrimination in presence of congestion externalities. Czerny and Zhang (2015) find that price discrimination by a monopoly airline to two classes of passengers (where, as in our case, demands are only interrelated though congestion) always reduces the aggregate quantity under linear
demands. The key difference lies in the properties of the derived demands, which can differ essentially with the final good demands. Czerny and Zhang (2015) assume that demands are such that the (final good) price is higher in the market where time valuations are higher and this implies that price discrimination cannot increase output. Although this seems adequate in their setting as, for example, one can think of business and leisure passengers, it is not necessarily appropriate for input prices. This is because the assumption that there is a strong market where consumers pay a higher price in equilibrium is not necessarily a good proxy for the relative magnitude of the input prices under price discrimination. It is possible that $w_B > w_A$ holds and that the equilibrium downstream price is higher in market $A$.

The effect of price discrimination on prices also extends Layson (1998). He shows, under linear demands, that for prices to move in the same direction when price discrimination is allowed, cross-price effects must be asymmetric and the firm’s marginal costs must be decreasing. In our setting, only asymmetric cross effects are required ($v_A \neq v_B$) as marginal costs are constant. The difference here is that if the (input) price rises in one market, the aggregate quantity does not necessarily decrease because of congestion effects: when the price in one market increases, the full price in the other market may decrease because of the decreased congestion costs. The results also extend Czerny and Zhang (2015) who find that price discrimination cannot reduce both prices, but only increase them. Again the difference is in their assumption on the relative magnitude of final prices.

The intuition of why price discrimination can reduce both prices is similar to the one provided by Layson (1998). In our model, an increase in the input price of one market increases the profitability of the other market, as congestion costs decrease. This increases the consumers’ willingness to pay and therefore the price that can be charged. Under uniform pricing, the marginal profit of the input provider in each market has a different sign. Consider that the marginal profit is negative for market $A$ under uniform pricing (consistent with $w_B > w_A$). If the marginal profit increases slowly towards zero, the decrease in price towards the optimally differentiated $w_A$ will be large. This large decrease may cause a large reduction in the profitability in market $B$, which was positive at uniform prices, and can make it negative at $\{w_A, w\}$. This will therefore cause a reduction also in the price in market $B$. This is what happens when the facility sets a higher input price in the market whose consumers have a lower value of time and time valuations are sufficiently different in that $v_A - 3 \cdot v_B > -4 \cdot P_B'/D'$. A similar explanation works for the case where both prices increase.

Which of the results in Proposition 1 is more likely to take place depends on the relation between time valuations and reservation prices. In our model, there are three sources of asymmetry between markets: time valuations, reservation prices and inverse demand slopes. All three are arguably correlated through (average) income: a higher income in market $A$ would explain a higher time valuation, and it would also imply that the reservation price is higher and the demand less sensitive to price changes. The sign of the output effect depends on whether $w_B > w_A$ holds or not. Using Lemma 1, we obtain that output is
more likely to increase with price discrimination when the ratio \( A_B/A_A \) is greater than the ratio of time valuations \( v_B/v_A \) and it will decrease if the asymmetry in reservation prices is similar to or higher than the asymmetry of time valuations. This, naturally, will depend on how the differences in income impacts the time valuations and reservation prices and it is ultimately a matter of empirical investigation. Second, the way in that prices change with price discrimination depend also on how asymmetric the time valuations are. Price discrimination is likely to move prices in the opposite direction when the ratio of time valuations \( v_B/v_A \) is not too low (higher than 1/3 is sufficient) and to change prices in the same direction when it is sufficiently low (\( v_B/v_A \) at least lower than 1/3). As explained above, this is because when they are sufficiently different \( (v_A - 3 \cdot v_B > -4 \cdot P_B'/D') \), the change in profitability in one market due to the change in the input price of the other is large.

The likelihood of \( v_A - 3 \cdot v_B > -4 \cdot P_B'/D' \) is somewhat difficult to assess. One way of casting light into its likelihood is by considering that the differences across markets are caused by differences in trip purpose. Koster et al. (2011), Kouwenhoven et al. (2014) and Shires and De Jong (2009) provide empirical evidence that the ratio of time valuations between business and other users in transport markets is not higher than 3. This suggests that \( v_A - 3 \cdot v_B > -4 \cdot P_B'/D' \) is a rather stringent condition when differences between markets are caused by differences in the proportion of business and other types of travelers. In that case it is more likely that input price discrimination increases the price in one market and decreases the price in the other. Which market faces the high price depends on the relation between time valuations and reservation prices as discussed earlier. If the ratio of demand intercepts is similar as the ratio of time valuations, for example, because income affects reservation prices and time valuations in a similar way, then price discrimination will increase the price in the market with high time valuations (market A). Only when the ratio \( A_B/A_A \) is greater than the ratio of time valuations \( v_B/v_A \) price discrimination can increase the input price in the low income market (market B).

### 3.4. Welfare analysis

A full characterization of the marginal welfare effect would be tedious in our case. First, unlike the case of final good markets, under the uniform pricing regime there is, in general, a misallocation of output between markets. This is because downstream firms charge a markup related to demand characteristics and time valuations, so that when the input price is uniform, the marginal willingness to pay is, generally, not the same in each market. To see this, consider the marginal change in total welfare as more discrimination is allowed using the same method as in the previous section:

\[
\frac{dW}{dt} = \frac{dq_A}{dt} \left[ (w_A - w) - q_A \cdot P_A' + q_A \cdot v_A \cdot D' \right] + \frac{dq_B}{dt} \left[ (w_B - w) - q_B \cdot P_B' + q_B \cdot v_B \cdot D' \right] + \frac{dQ}{dt} \cdot \left[ w - [q_A \cdot v_A + q_B \cdot v_B] \cdot D' \right],
\]

where the first two terms in square brackets are the final good prices \( (w_i - q_i \cdot P_i' + q_i \cdot v_i \cdot D') \) minus the uniform input price input price set by the facility \( (w) \), and the third bracketed
term is the difference between the uniform input price and the marginal external congestion cost. When the input prices are uniform and equal to $w$, there is still a misallocation effect unless the sum of the demand related markup and the internalized congestion is the same in both markets $(-q_A \cdot P'_A + q_B \cdot v_A \cdot D' = -q_B \cdot P'_B + q_B \cdot v_B \cdot D')$.

Second, as Czerny and Zhang (2015) point out, the presence of congestion externalities gives rise to an effect that works in the opposite direction as the output effect on welfare. Thus, welfare can increase when output is decreased by price discrimination. This section provides a partial characterization of the effect of price discrimination on welfare by deriving sufficient conditions for welfare improvement and deterioration. Rearranging Eq. (14), we get:

$$
\frac{dW}{dt} = \frac{dq_A}{dt} \cdot \left[w_A - \left[q_A \cdot P'_A + q_B \cdot v_B \cdot D'\right]\right] + \frac{dq_B}{dt} \cdot \left[w_A + t - \left[q_B \cdot P'_B + q_A \cdot v_A \cdot D'\right]\right],
$$

(15)

where the terms in square brackets multiplying the marginal quantity changes are the difference between the input price set by the facility and the socially optimal input price.

The welfare analysis can be divided in two cases, namely when price discrimination changes both quantities in the same direction (both either rise or fall) and when price discrimination increases the quantity in one market and it decreases it in the other. We first focus in the latter case. Opposite changes in demand due to price discrimination are a consequence of opposite changes in prices. As discussed in Proposition 1, this happens when time valuations are not too different and, thus, the effect of a price change in one market on the marginal profitability of the other is not large enough to provide incentives to increase or decrease both input prices. In this case, the output increases in the market where the input price decreases and it decreases in the other market. We provide sufficient conditions for welfare improvement when aggregate output increases and for welfare deterioration when aggregate output decreases. As shown in Proposition 1, the sign of the output effect depends on whether $w_B > w_A$ holds or not, which depends on whether $A_B / A_A > \lambda_1$ holds or not. First, if demands are such that $w_B > w_A$ ($A_B / A_A > \lambda_1$), the aggregate output increases and, therefore, the quantity decrease in market $B$ is lower than the increase in market $A$. As a consequence, from Eq. (15), if the difference in actual and socially optimal input price is positive in market $A$ and higher than in market $B$ for all values of $t$, then price discrimination necessarily increases welfare. Conversely, if $w_B < w_A$ holds ($A_B / A_A < \lambda_1$), the aggregate output decreases and the quantity decrease in market $A$ is higher than the increase in market $B$. Therefore, if the difference in actual and socially optimal input price is always positive and higher in market $A$, welfare decreases. The conditions for this are summarized in the following proposition:

**Proposition 2.** When time valuations are similar in that $v_A - 3 \cdot v_B < -4 \cdot P'_B / D'$, the quantities change in opposite directions with price discrimination and:

(i) Price discrimination increases welfare if:

$$
\lambda_1 < \frac{A_B}{A_A} < \lambda_2 = \frac{12 \cdot P'_A \cdot P'_B + 10 \cdot v_A \cdot v_B \cdot D'^2 + 2 \cdot v_A^2 \cdot D'^2 + 3 \cdot v_B \cdot D' \cdot [-4P'_A - P'_B] - 11 \cdot P'_B \cdot v_A \cdot D'}{12 \cdot P'_A \cdot P'_B + 10 \cdot v_A \cdot v_B \cdot D'^2 + 2 \cdot v_A^2 \cdot D'^2 + 3 \cdot v_B \cdot D' \cdot [-4P'_A - P'_B] - 11 \cdot P'_A \cdot v_B \cdot D'}.
$$
(ii) Price discrimination decreases welfare if:

\[ \frac{A_B}{A_A} < \min[\lambda_1, \lambda_3], \text{ where } \lambda_3 = \frac{-P'_A + v_A \cdot D'}{[4P'_A + v_A \cdot D' + 5v_B \cdot D'] - 4P'_B + v_B \cdot D'} \]

Let us first discuss part (i), where \( w_B > w_A \) holds and aggregate quantity increases. The reason why welfare increases is that the benefit in the market \( A \) from a decreased input price and increased quantity is larger than the loss in market \( B \), where the opposite happens. Therefore, it follows that demand in market \( B \) cannot be significantly larger than in market \( A \) for this to hold. This is why an upper bound on \( \frac{A_B}{A_A} \) is needed. It is expected that the market with high time valuations is also the market with low demand price sensitivity, so that \( v_A > v_B \) and \( -P'_A > -P'_B \) hold. In this case, it is straightforward to show that \( \lambda_2 < 1 \).

In addition, the interval \([\lambda_1, \lambda_2]\) is non-empty when the ratio of the inverse demand’s slopes is less than the ratio of time valuations, i.e. \( \frac{v_B}{v_A} < \left| \frac{-P'_B}{-P'_A} \right| \). Thus, price discrimination is likely to increase welfare when time valuations are similar (\( \frac{v_B}{v_A} > 1/3 \) is sufficient) and the price sensitivities as well as the reservation prices are more similar.

The second part of Proposition 2 is intuitive: if \( \frac{A_B}{A_A} \) is lower than \( \lambda_1 \), price discrimination increases the price in the market \( A \), which is the market with a higher reservation price and with higher time valuations. This, not surprisingly, is likely to reduce welfare. If \( -P'_A > -P'_B \) holds, which is expected to hold if the difference of time valuations across markets is due to differences in income, \( \lambda_3 > \lambda_1 \) holds and therefore \( \frac{A_B}{A_A} < \lambda_1 \) is a sufficient condition for welfare deterioration. \( \lambda_3 \) is only part of the necessary condition in the case where \( -P'_A < -P'_B \) and it is sufficient for the welfare loss in the market \( A \) to be higher than the gain in \( B \). As a result, if time valuations are similar (a ratio higher than 1/3 is sufficient), the market with high time valuations is also the market with lower demand sensitivity to price changes and the ratio of demand intercepts is similar to or lower than the ratio of time valuations, price discrimination will decrease welfare.

The welfare analysis when price discrimination changes both quantities in the same direction is in Appendix A. We choose not to discuss it here because the conditions that make price discrimination to increase or decrease both quantities are rather stringent and not very informative. We show that both prices moving in the same direction is not sufficient for both quantities to move in the same direction because of congestion effects. A change in demand in one market has an impact on the full price of the other market and, if cross congestion effects are not low, this can overturn the effect of the own input price change. The conditions that make quantities to either rise or fall involve an upper and lower bound on the ratio of time valuations and also a restriction on the relationship between time valuations and demand slopes. Moreover, the time valuation in market \( A \) has to be more than 5 times larger than in market \( B \), which, as we argue above, seems to be unrealistic in transport markets. Nevertheless, when price discrimination increases the quantity in both markets it increases consumer surplus in both markets and total welfare. The reverse may also happen and price discrimination can decrease both quantities, decrease welfare and consumer surplus.

The results of this section show benefits from input price discrimination in the presence
of negative consumption externalities and that price discrimination can increase consumer surplus. Importantly, the benefits are found in a setting where, in absence of externalities, price discrimination yields lower social welfare. In addition, the conditions for welfare improvement depend strongly on the absolute and relative value of the congestion effects. This suggests that the efficiency of a pricing policy can differ with the level of congestion of the facility even if everything else is invariant (e.g. through different capacity of the facility).

A natural expectation when the differences in time valuations arise from differences in income across markets is that the market with high time valuations \( A \) also exhibits lower demand sensitivity to price changes \( -P_A' > -P_B' \) and a higher reservation price \( A_A > A_B \). In this case, it follows from Propositions 1 and 2 that price discrimination is more likely to decrease welfare when the asymmetry of the reservation prices is similar to or higher than the asymmetry in time valuations (i.e. when \( A_B/A_A \leq v_B/v_A \)). This is because under linear demands and congestion the market with a higher demand intercept is the less elastic market. It also follows that for price discrimination to increase welfare, the ratio of reservation prices \( A_B/A_A \) needs to be higher than the ratio of time valuations. In addition, when time valuations are similar in that \( v_B/v_A \geq 1/3 \) holds, price discrimination is more likely to increase welfare when the ratio \( A_B/A_A \) is, besides being larger than \( v_B/v_A \), not too high. For example, when \( 1/3 < v_B/v_A < 1/2 \), price discrimination decreases welfare when \( A_B/A_A < 2/3 \) and it can increase welfare if \( 2/3 < A_B/A_A < 1 \).

In the following section we analyze price discrimination by a public facility with the aim of comparing the welfare results and shed light on when a broad ban on price discrimination is desirable.

4. Public facility

We now study a public facility that maximizes domestic welfare. If the facility were maximizing total welfare, allowing price discrimination would always be optimal and the analysis would be trivial. We introduce a source of divergence from total welfare maximization, namely that consumers and firms may be foreign. Among the many possible domestic-foreign structures we consider the case where market \( A \) is fully domestic (passengers and firm \( A \) are domestic) and the firm \( B \) together with a fraction of the passengers in market \( B \) are foreign. The assumption that the market with higher time valuations is the domestic market is, we believe, a realistic assumption if the differences in income across markets are a consequence of differences in trip purpose, as business travel is more frequent in domestic destinations than in international travel. For example, in 2012, the share of business trips was 20%, 30% and 190% higher in domestic destinations than international destinations at London City (LCY), London Heathrow (LHR) and Manchester (MAN) airports respectively (CAA, 2012). In 2011 in Los Angeles International Airport (LAX), the share of business trips was 90% higher in U.S. destinations than in international
destinations (Unison Consulting, 2011).\textsuperscript{10}

For simplicity we assume that the fraction of foreign passengers in market $B$ is 1/2 and, therefore, the public facility maximizes the sum of its profit, firm $A$’s profit, the consumer surplus in market $A$ and one half of the consumer surplus in market $B$:

\[
W_D = \left[ \int_0^{q_A} P_A(x)dx - v_A \cdot q_A \cdot D(Q) \right] + \frac{1}{2} \left[ \int_0^{q_B} P_B(x)dx - q_B \cdot P_B(q_B) \right] + \left[ w_B \cdot q_B \right].
\]

(16)

where the first term in square brackets is total welfare in market $A$ (the sum of the consumer surplus, firm $A$’s profit and airport revenues from market $A$), the second term in square brackets is the consumer surplus in market $B$ and the third term is the airport’s revenue from market $B$.

The incentive to price discriminate is to capture part of the foreign firm’s profit and stimulate domestic production. The model can easily be extended to cases where a different share of consumer surplus is taken into account by the facility or to cases where there are foreign passengers in both markets, but results do not change in any significant way. What matters is that there is a clear incentive to reduce the price in one market in detriment of the other, and not so much which is the mechanism that provides this incentive.

\subsection{4.1. Price discrimination}

The first-order conditions of maximizing $W_D$ with respect to both input prices lead to the input prices $w_A$ and $w_B$ (see Appendix B for the prices and all derivations of the results in this section). Here, as in the previous section, we present the pricing rules:

\[
w_A = q_A \cdot P'_A + q_B \cdot v_B \cdot D',
\]

(17)

\[
w_B = 2 \cdot q_B \cdot \left[ -\frac{3}{4} \cdot P'_B + v_B \cdot D' \right] + q_A \cdot v_A \cdot D'.
\]

(18)

The input price for the domestic firm is a subsidy equal to the downstream markup ($q_A \cdot P'_A < 0$) and the marginal congestion cost that is not internalized by firm $A$ ($q_B \cdot v_B \cdot D'$).\textsuperscript{11} This is the first-best pricing rule, as it makes the final price in the market equal to the marginal social cost (see Eq. (4)). The price in the foreign market is the sum of a market power markup and the marginal congestion cost that is not being internalized. The public facility does not subsidize the foreign firm, but the markup is lower than in the private case, as the consumer surplus in this market is partially taken into account because a fraction of the consumers are domestic.

\textsuperscript{10}Our assumption may be less realistic for air transportation in high income countries with small domestic markets, such as the Netherlands or Switzerland. In those cases, our model may be representative of other transportation markets where congestible facilities provide an input to downstream firms, such as rail transportation.

\textsuperscript{11}The facility charges the externality imposed on the foreign market because it is profit maximizing to do so (see the pricing rule of the private facility in Eq. (6)) and not because of consumer surplus considerations.
From comparing the pricing rules above, it follows that \( w_B > w_A \) always holds in this case, a result of the assumed domestic-foreign structure. This captures the usual argument to enforce uniform pricing by a public supplier that it protects consumers of foreign markets. In addition, \( w_B \) is always positive and the sign of \( w_A \) is ambiguous and depends on whether the inefficiency due to downstream market power is larger or smaller than the inefficiency due to the congestion externality.

4.2. Uniform pricing

When the facility is restricted to charge the same input price to both firms, we obtain the following pricing rule:

\[
\begin{align*}
\frac{dQ}{dt} &= 2 \left[ -P'_A + v_A \cdot D' \right] - v_A \cdot D' \\
&= 2 \left[ -P'_A + v_A \cdot D' \right] - v_A \cdot D' \\
&= 2 \left[ -P'_A + v_A \cdot D' \right] - v_A \cdot D' \\
&= 2 \left[ -P'_A + v_A \cdot D' \right] - v_A \cdot D'
\end{align*}
\]

The pricing rule in Eq. (19) includes a weighted sum of the subsidy for firm \( A \) and the markup for firm \( B \) present in the discriminating prices. It also includes a weighted sum of the market-specific marginal congestion cost that are also part of the differentiated input prices. We elaborate in the following section on the relation between the uniform and the discriminating input prices and show that the uniform price is a weighted average of the discriminatory prices.

4.3. The effects of price discrimination on prices and output

Using the same price difference constraint method as in Section 3, we obtain the following results regarding the marginal effect of price discrimination on the aggregate output and on input prices:

\[
\begin{align*}
\frac{dQ}{dt} &= \frac{2 \left[ -P'_A + v_A \cdot D' \right] - 5 \cdot P'_B + 4 \cdot v_B - v_A}{\Omega_3} \\
&= \frac{2 \left[ -P'_A + v_A \cdot D' \right] - 5 \cdot P'_B + 4 \cdot v_B - v_A}{\Omega_3}
\end{align*}
\]

where \( \Omega_3 \) is a positive constant. The results that follow from Eqs. (20)–(22) are summarized in the following proposition.
Proposition 3. Price discrimination by a public facility:

(i) Increases the aggregate output if time valuations are sufficiently similar in that \( v_A - 3 \cdot v_B < -4 \cdot P'_B/D' \).

(ii) Decreases the input price in the market served by the domestic firm.

(iii) Increases the input price in the market served by the foreign firm.

This is intuitive, when the facility is allowed to price discriminate it reduces the price in the market served by the domestic firm and raises the price to the foreign firm to capture part of its profit. When the condition (i) in Proposition 3 holds, the output increase in the market served by the domestic firm is larger than the decrease in the market served by the foreign firm.

4.4. Welfare analysis

Unlike in the case of a private facility, we can analyze the welfare effect directly as opposed to using the price difference constraint method used in Section 3. Recall that in this section we look at how total welfare changes when a facility that maximizes domestic welfare is allowed to differentiate prices. The main result of the analysis is summarized in the following proposition.

Proposition 4. Price discrimination by a public facility increases total welfare if, and only, if \( A_B/A_A < \lambda_4 \), and it decreases total welfare when \( A_B/A_A > \lambda_4 \).

Where \( \lambda_4 \) is a fraction whose numerator and denominator are a function of the demand sensitivity parameters \( (P'_A, P'_B) \) and of the congestion effects \( (v_A \cdot D', v_B \cdot D') \) in a similar way as \( \lambda_1 \) and \( \lambda_2 \). However, both the numerator as well as the denominator of \( \lambda_4 \) are polynomials of degree 7, so we omit the expression here (see Appendix B).

From a total welfare standpoint, price discrimination leads to a welfare loss in the market served by the foreign firm \( (B) \), as the price moves away from the marginal social cost, and to a welfare gain in the market served by the domestic firm \( (A) \), as the price moves towards marginal social cost. To obtain intuition for the result in Proposition 4 consider the case where there are no congestion effects \( (v_A = v_B = 0) \). In this case, \( \lambda_4 \) is a function only of the demand slopes and if the inverse demand is steeper in market \( A \) (i.e. \( -P'_A > -P'_B \)) \( \lambda_4 > 1 \) holds. Therefore, in absence of congestion, if the foreign market is more elastic at uniform prices \( (A_A > A_B) \) and it also has a higher sensitivity to price changes, total welfare increases with price discrimination. This result is natural, price discrimination raises the price in market \( B \), which is the more elastic and the more price sensitive, so the welfare losses due to the double marginalization are limited compared to the gains of pricing the domestic market at marginal social cost. In the general case, congestion effects come into play and patterns are complex. Increased demand can have negative effects and the cross-effects that resemble substitution may change the conclusions. Nevertheless, the result in Proposition 4 that there is an upper bound for \( A_B/A_A \) for total welfare improvement is intuitive. Price discrimination is more likely to increase total welfare when the foreign market –where the input price is raised– is relatively more elastic and not too large. For
example, when $v_B/v_A = 1/3$, the lowest ratio of time valuations that ensures that output increases with price discrimination, $\lambda_4 > 1/3$ holds. As a result, price discrimination increases total welfare when the ratio of the demand intercepts is the same as or lower than the ratio of time valuations. When the time valuations are not more asymmetric than the reservation prices, the congestion effects do not overturn the results obtained in absence of congestion. Importantly, this is in sharp contrast with the results for a private facility where price discrimination is likely to reduce total welfare when the two ratios ($v_B/v_A$ and $A_B/A_A$) are similar.\textsuperscript{12} In the following section we argue that $\lambda_4 > v_B/v_A$ is likely to hold more generally, so the conclusion that price discrimination by a public facility increases total welfare when the ratio of the reservation prices is similar to the ratio of time valuations is not restricted to the particular case of $v_B/v_A = 1/3$.

In the following section we also compare the effect of allowing price discrimination on total welfare under both ownership forms in more detail and study whether a broad ban, that covers facilities with different ownership forms, is desirable.

5. Comparison of the welfare effect under private and public ownership

To compare the welfare effect of price discrimination we focus on what we believe is the most realistic setting: a case in which, in market $A$, the time valuation is higher, the demand sensitivity to price changes is lower and the reservation price is higher than in market $B$. This is a natural expectation if the differences across markets are caused by differences in income and average income is higher in market $A$. Consequently, throughout this section we assume that $v_A > v_B$, $-P_A' > -P_B'$ and $A_A > A_B$ hold.

We also limit the comparison to the case where time valuations are similar in that $v_B/v_A \geq 1/3$ holds, which ensures that price discrimination by a private facility changes prices in opposite directions (see Proposition 1). We do not compare the welfare effect of price discrimination when price discrimination by a private facility either increases or decreases the output in each market, because the sufficient conditions for the quantities to move in the same direction are too stringent for the case of a public facility. That is, the parameter region where each firm’s output is positive in equilibrium under price discrimination by a public facility is very limited. Moreover, estimations of time valuations suggest that the condition $v_B/v_A \geq 1/3$ is realistic. Koster et al. (2011) and Kouwenhoven et al. (2014) estimate the value of access time, the value of schedule delay and the value of travel time savings for business and other travel purposes in Dutch air transport passengers. They find that the ratio between other purposes and business time valuations is higher than 0.5 in all cases. This is a lower bound on the ratio of time valuation between markets, if differences between markets are a consequence of different composition of business and

\textsuperscript{12}The results in Proposition 4 are also valid when the market served by the foreign firm is also the one where consumers have higher time valuations ($v_B > v_A$). In this case, $\lambda_4 > 1$ holds and price discrimination always increases total welfare when the inverse demand intercept in the market served by the domestic firm is larger than or of the same size as the one served by the foreign firm.
other travelers. A meta-analysis covering 30 countries and 77 studies that estimate values of travel time savings in different modes by Shires and De Jong (2009) find that in average, the ratio between time valuations of commuting travelers and business travelers is 0.4. They also report that the ratio between other purposes and commute is on average 0.84, which implies that the lowest ratio is 0.336 (between non commuting and business travelers). This evidence also supports the relevance of the case we study in this section.

When \( \frac{v_B}{v_A} \geq \frac{1}{3} \) holds, price discrimination by a public facility increases aggregate output, increases the input price in market \( B \) and decreases the input price in market \( A \). Price discrimination by a private facility also increases aggregate output, raises the price in market \( B \) and decreases the price in market \( A \) if \( w_B > w_A \) holds \( (A_B/A_A > \lambda_1) \). Conversely, if \( w_B < w_A \) holds \( (A_B/A_A < \lambda_1) \) it decreases aggregate output, the price in market \( B \) falls and the price in market \( A \) increases. The relevant comparison is between Propositions 2 and 4 and the main results are summarized in the following proposition (the proof follows directly from the propositions):

**Proposition 5.** When time valuations are sufficiently similar \( (v_B/v_A \geq 1/3) \), the market with higher time valuations is also the market with lower demand sensitivity to price changes and it is the domestic market:

(i) A ban on price discrimination is desirable only for a private facility if \( \frac{A_B}{A_A} < \min[\lambda_1, \lambda_4] \)
(ii) A broad ban on price discrimination is desirable if \( \lambda_4 < \frac{A_B}{A_A} < \lambda_1 \)
(iii) A ban on price discrimination is desirable only for a public facility if \( \max[\lambda_1, \lambda_4] < \frac{A_B}{A_A} < \lambda_2 \)

This summary allows for shedding light on the desirability of a broad ban on price discrimination. First, if the reservation price in the domestic market is significantly larger than the reservation price in the foreign market \( (A_A >> A_B) \), a ban that covers both ownership forms may not be desirable. In this case it is socially optimal to have a public facility differentiating prices as the potential benefits from increased domestic production in market \( A \) are large compared to the losses in market \( B \). This is because \( A_A >> A_B \) implies that market \( B \) is much more elastic at uniform prices and the welfare losses are limited because the markup is limited.

Second, when time valuations are not too close to each other, i.e. \( 1/3 < v_B/v_A \leq 4/5 \), and the asymmetry in time valuations, in reservation prices and in demand sensitivity to price changes is similar, a broad ban on price discrimination is not desirable. This is because \( \lambda_1 > v_B/v_A \) holds regardless of other parameters and when \( 1/3 < v_B/v_A < 4/5 \) and \( -P'_B/ - P'_A > 1/10 \) hold, \( \lambda_4 > v_B/v_A \) holds. Therefore, whenever the asymmetry in reservation prices is similar to the asymmetry in time valuations \( (A_B/A_A \approx v_B/v_A) \) or higher \( (A_B/A_A < v_B/v_A) \), allowing a public facility to differentiate prices raises total welfare and a broad ban cannot be the welfare maximizing pricing policy. The intuition is similar as in the previous case. In absence of congestion \( A_B/A_A < 1 \) and \( -P'_A > - P'_B \) are sufficient for price discrimination by a public supplier to be welfare improving. As explained in the previous section, this is because the higher price sensitivity and the higher elasticity
at uniform prices of market $B$ limit the markup and welfare losses. As argued in Section 4, congestion effects may overturn this. However, if the time valuations are as symmetric or more symmetric than the reservation prices, it is less likely that the welfare effect is overturned. This is why when $A_B/A_A \leq v_B/v_A$, it is efficient to allow price discrimination by a domestic welfare maximizing facility.

Third, the results above suggest that when time valuations are not too similar (i.e. $1/3 < v_B/v_A \leq 4/5$) a broad ban may be desirable if the asymmetry in reservation prices is lower than the asymmetry in time valuations and if the reservation prices are not too close to each other. This is because $\lambda_4 < A_B/A_A < \lambda_1$, the condition (ii) of Proposition 5, is needed, but $\lambda_4 > v_B/v_A$ and $1 > \lambda_1 > v_B/v_A$ hold. However, $\lambda_4 < \lambda_1$ does not hold globally. In the limit where $v_A = v_B = 0$ and when $v_A \to \infty$ it does not hold so a broad ban on price discrimination may not be desirable, but it may hold for intermediate values of $v_A$. We use numerical examples below to show that this may occur.

Fourth, when time valuations are such that $4/5 < v_B/v_A < 1$ holds, numerical results show that the intuition provided above also holds as long as the asymmetry in inverse demand slopes is not significantly higher than the asymmetry in time valuations. That is, when the ratio of time valuations, inverse demand slopes and reservation prices are similar, a ban on price discrimination is not desirable for both ownership forms also in the case where $4/5 < v_B/v_A < 1$. For example, if $v_B/v_A = 0.9$, $\lambda_4 > v_B/v_A$ holds for all values of $P_B' - P_A'$ higher than 0.22. In this case, $A_B/A_A \leq 0.9$ is sufficient for ban on price discrimination not to be welfare enhancing for both ownership forms. Again, if the reservations prices are more similar than time valuations, a broad ban may be desirable. The reason is the same as for the case $1/3 < v_B/v_A \leq 4/5$. Price discrimination by a public supplier is likely to increase welfare when cross congestion effects are not more asymmetric than the reservation prices and inverse demand slopes.

Finally, from the sufficient conditions derived in Sections 3 and 4 we cannot assess the desirability of a broad ban when the reservations prices are very similar across markets (i.e. when $\lambda_1 < A_B/A_A$ and $\lambda_2 < A_B/A_A \leq 1$). Nevertheless, as $\lambda_1 > v_B/v_A$ holds, the main conclusion that under similar asymmetry across markets a broad ban may not be desirable is not affected by this lack of sufficient conditions for the welfare change.

We complement the results of Proposition 5 and the intuition provided with numerical examples. The numerical analysis in Figure 1 shows the values of $\lambda_1$, $\lambda_2$ and $\lambda_4$ for different parameter values. In each panel the values of the inverse demand slopes $-P_B'$ and $-P_A'$ are fixed and two values of time valuation ratios ($v_B/v_A$) are studied. What varies in each figure is the value of time in market $A$, $v_A$. The cases where a broad ban on price discrimination is desirable for both ownership forms are highlighted in gray (the condition (ii) in Proposition 5 holds). In the white areas, a ban on input price discrimination fails to be efficient for, at least, one ownership form. The numerical analysis reinforces the discussion above that a

\footnote{The values of $-P_B'/-P_A'$ that ensure that $\lambda_4 > v_B/v_A$ hold are 0.13, 0.22 and 0.39 for $v_B/v_A$ equal to 0.85, 0.9 and 0.95 respectively.}
ratio of reservation prices $A_B/A_A$ equal to or lower than the ratio of time valuations is a sufficient condition for a broad ban not to be welfare enhancing for both ownership forms. It also shows that $A_B/A_A$ can be higher than $v_B/v_A$ and the result still holds. For the studied parameter range, when the ratio of time valuations equals 0.5 a broad ban may not be desirable when the ratio of reservation prices is lower than 0.8. When $v_B/v_A = 0.85$, $A_B/A_A < 0.91$ ensures the inefficiency of a ban on price discrimination for at least one ownership form. The comparison between Figures 1a and 1b also confirms that condition (ii) of Proposition 5 ($\lambda_4 < A_B/A_A < \lambda_1$) does not hold globally. In our numerical example, a broad ban can be desirable for the case where the ratio of inverse demand slopes equals $1/3$ and not when it is $2/3$. This suggests that the feasibility of the condition is more closely related with the absolute value of $-P'_B/ -P'_A$ rather than its relative value with respect to $v_B/v_A$, as in Figure 1b $v_B/v_A > -P'_B/ -P'_A$ holds in one case and $v_B/v_A < -P'_B/ -P'_A$ in the other and in neither of them the condition holds. The results in Figure 1a also confirm that for relatively low and high values of $v_A$ a ban cannot be desirable for both types of facilities and it shows that for intermediate values it may be desirable when reservation prices are similar but not too close to each other. In our example, this happens for values of $A_B/A_A$ between 0.8 and 0.9 when the ratio of time valuations is $1/2$ and for values of $A_B/A_A$ between 0.91 and 0.97 when the ratio of time valuations is 0.85.

Note that for the comparisons of this section we have used sufficient conditions for welfare improvement and deterioration under price discrimination by a private facility instead of comparing the actual effect. Therefore, the regions in which a ban on price discrimination is the socially optimal policy for both ownership forms may be larger than the shaded

![Figure 1](image-url)

Figure 1: Sufficient conditions for welfare improvement and deterioration under price discrimination. In the gray areas an ban on input price discrimination is desirable for both ownership forms. In the white areas a ban on input price discrimination fails to be efficient for at least one ownership form. Parameter values: $P'_A = -1$, $P'_B = -1/3$, $D' = 1$ in Figure 1a and $P'_A = -1$, $P'_B = -2/3$, $D' = 1$ in Figure 1b.
6. Robustness: downstream first-degree price discrimination

In this section we analyze the robustness of the conclusions drawn in the previous section from comparing the effect on total welfare of third-degree price discrimination by a public and a private facility. We study how the main results change when downstream firms apply first-degree price discrimination. This is a theoretical extreme that is useful also to study a situation where there is no downstream inefficiency due to market power and works as a proxy for a perfectly competitive downstream market.

Downstream firms that perfectly discriminate consumers set a unit price equal to the marginal cost, which is the input price \( w_i \) plus the marginal congestion cost that is internal to the firm \( q_i \cdot v_i \cdot D'(Q) \), and ask for a premium equal to the surplus of each individual (their willingness to pay net of the experienced delays). This changes the derived demands faced by the input provider which are now a result from the following pricing rules:

\[
P_i(q_i) - v_i \cdot D(Q) = w_i + q_i \cdot v_i \cdot D'(Q).
\]

Following the same methodology as in Sections 3 and 4, it is possible to derive similar sufficient conditions for welfare improvement and deterioration under third-degree input price discrimination for both ownership forms. The aim of this extension is to analyze how the results in Proposition 5 change. Let \( \lambda'_1 \) be the analogous boundary to \( \lambda_i \) derived in Sections 3 and 4. Proposition 5 can be restated in the following way:

**Proposition 6.** When downstream firms can perfectly discriminate consumers, time valuations are sufficiently similar \((v_B/v_A \geq 1/3)\), the market with higher time valuations is also the market with lower demand sensitivity to price changes and it is the domestic market:

- (i) A ban on price discrimination is desirable only for a private facility if \( \frac{A_B}{A_A} < \min \{\lambda'_1, \lambda'_3, \lambda'_4\} \)

- (ii) A broad ban on price discrimination is desirable if \( \lambda'_4 < \frac{A_B}{A_A} < \min[\lambda'_1, \lambda'_3] \)

- (iii) A ban on price discrimination is desirable only for a public facility if \( \max[\lambda'_1, \lambda'_4] < \frac{A_B}{A_A} < \lambda'_2 \)

A difference with respect to Proposition 5 is the presence of \( \lambda'_3 \). When price discrimination by a private facility decreases aggregate output, \( \lambda'_3 \) is the upper bound for \( A_B/A_A \) such that the difference in actual and socially optimal input price is higher in market A than in market B under input uniform pricing. This is a sufficient condition for the loss due to the output contraction in market A to be larger than the benefit from increased production in market B. When demand in market A is less price-sensitive than in market B and downstream firms do not price discriminate, the resulting markups ensure that the condition is satisfied and \( \lambda'_3 \) becomes irrelevant. As under downstream perfect price discrimination the unit price is the marginal cost, the condition \( A_B/A_A < \lambda'_3 \) is needed again. Phrased differently, in absence of downstream markups related to the price sensitivity, the
difference between actual and socially optimal input price can be lower in market A than in B under input uniform pricing.

One of our main results is that if the asymmetry in reservation prices, the asymmetry in time valuations and the asymmetry in demand sensitivity to price changes are similar, a broad ban on price discrimination may not be desirable because it is optimal to allow a public facility to price discriminate firms. Under downstream first-degree price discrimination this is also the case if, in addition, the congestion effects are not too low. This is because \( \lambda'_{4} \geq v_{B}/v_{A} \) does not hold globally.\(^{14}\) To see why, first consider that there is no congestion (i.e. \( v_{A} = v_{B} = 0 \)). As there is no downstream market power inefficiency due to the perfect discrimination, in absence of negative consumption externalities, the socially optimal input prices are equal to zero. It can be shown that in this case, price discrimination by a public facility is always welfare decreasing (\( \lambda_{4} = 0 \) when \( v_{A} = v_{B} = 0 \)), something that does not hold when there is downstream inefficiency due to market power exertion. In the other extreme, in the limit where \( v_{A} \) goes to infinity, \( \lambda'_{4} > v_{B}/v_{A} \) and our main result holds under downstream perfect price discrimination. This implies that if congestion effects are sufficiently high, \( \lambda'_{4} > v_{B}/v_{A} \) holds and the main welfare result also holds. Therefore, when the asymmetry in reservation prices, the asymmetry in time valuations and the asymmetry in demand sensitivity to price changes are similar across markets and downstream firms perfectly price discriminate, there is always a level of congestion that makes a ban on price discrimination inefficient for at least one ownership form.

The results of this section show that allowing input providers to price discriminate can increase total welfare even in the extreme case of downstream perfect price discrimination. Therefore, the benefits from input price discrimination do not rely on the presence of downstream market power inefficiencies and it can be expected that for imperfect downstream price discrimination the results are closer to those in the previous sections.

To conclude this section we briefly analyze two numerical examples. Figures 2a and 2b summarize the sufficient conditions for welfare improvement and deterioration when price discrimination is allowed for both ownership forms. We set the parameter values to the values used for the numerical analysis in Figure 1b, where the sufficient condition for a broad ban to be desirable was never satisfied. The gray areas display the parameter regions in which a ban on price discrimination is socially optimal for a private as well as for a public facility. Both figures show that, as discussed above, a broad ban on price discrimination can be the optimal policy if congestion effects are not too high. This suggest that the absence of downstream market power inefficiencies enhances the performance of a ban on price discrimination. Figures 2a and 2b also reveal that when the asymmetry in reservation prices is similar to the asymmetry in time valuations (\( A_{B}/A_{A} = 0.5 \) in

\(^{14}\lambda'_{4} > v_{B}/v_{A} \) always holds, \( \lambda'_{3} > v_{B}/v_{A} \) holds if \(-P_{B}' / -P_{A}' \geq v_{B}/v_{A} \) and when \(-P_{B}' / -P_{A}' < v_{B}/v_{A} \) holds, \( \lambda'_{3} > v_{B}/v_{A} \) holds if \(-P_{B}' / -P_{A}' \) is not too low compared to \( v_{B}/v_{A} \). For example, when \( v_{B}/v_{A} \) is equal to \( 1/3, 1/2, 2/3 \) and \( 0.9 \), \(-P_{B}' / -P_{A}' \) needs to be higher than \( 1/16, 1/6, 1/3 \) and \( 0.75 \) respectively for \( \lambda'_{3} > v_{B}/v_{A} \) to hold.
Figure 2a and $A_B/A_A = 0.85$ in Figure 2b), the extent to which a broad ban on price discrimination may be undesirable is large for the studied parameterization. Finally, the examples suggest that a broad ban is more likely to be desirable, just as in the previous analysis, when the asymmetry of reservation prices is lower than the asymmetry of time valuations ($A_B/A_A > v_B/v_A$).

![Figure 2](image)

Figure 2: Sufficient conditions for welfare improvement and deterioration under price discrimination. In the gray areas an ban on input price discrimination is desirable for both ownership forms. In the white areas a ban on input price discrimination fails to be efficient for at least one ownership form. Parameter values: $P_A' = -1, P_B' = -2/3, D' = 1$.

7. Conclusions

This paper has shown how the presence of congestion externalities influences the effects of input third-degree price discrimination. Our framework considers an upstream monopoly facility that can be private or public (domestic welfare maximizer) that sells an input to private downstream firms that operate in different markets as monopolists. The presence of downstream within- and cross-market negative externalities makes all demands interrelated in a way that is similar to the case where downstream firms offer substitute products. Using a stylized model with linear demands, we show that aggregate output can increase, all prices can decrease and welfare can increase when discrimination is allowed; although the opposite results are also possible. This is found in a setting in which in absence of congestion, price discrimination by a private input provider leads to welfare deterioration and constant aggregate output. The results of the paper suggest that the presence of congestion externalities enlarges the extent to which input price discrimination by a private facility is desirable from a welfare standpoint.
We have also analyzed the effects of price discrimination when the supplier maximizes domestic welfare, a common ownership form of transport facilities, and we have compared the welfare effects of price discrimination under private and public ownership. We have characterized the conditions that make the welfare maximizing pricing regime to be different for different ownership forms of the facility. Although the patterns are complex, the main insights are that when the asymmetry in time valuations, reservation prices and price-sensitivity of demand is similar across markets, a ban on price discrimination is not socially optimal for, at least, one type of facility. A broad ban may be desirable when the reservation prices in both markets are significantly more similar than the time valuations. Therefore, the results of this paper suggest that a broad ban on price discrimination to transport facilities that cover multiple ownership forms, such as the EU Airport Charges directive (2009/12/EC) and the World Trade Organization’s General Agreement on Tariffs and Trade (GATT), may have to be revised. This is especially relevant in the light of the increasing practice of (partially) privatizing transport facilities.

The extent to which the conditions that make the socially optimal pricing regime to be different between public and private ownership hold is an important avenue for empirical future research. The consideration of competition, cost regulations and network effects are also natural extensions for future research. However, as the divergence of the socially optimal policy has been found in the simple framework considered here, it is unlikely to disappear when complexity is added. Finally, the analysis has relied on linear functional forms. Extending the investigation considering other demand and cost functions is a natural and important avenue for future research. Analyzing demand functions with adjusted concavities that make price discrimination by a private facility to decrease output in absence of congestion is of particular relevance to check the robustness of the result that the presence of congestion enhances the performance of input price discrimination.

References


**Appendix A. Calculations and proofs for Section 3**

**Derived demands**

Solving simultaneously the first-order conditions of both downstream firms (see Eq. (2)) and denoting $A_i$ the inverse demand intercept in market $i$, we obtain the derived demands:

\[
\begin{align*}
q_A(w_A, w_B) &= \frac{2 \cdot [-P'_B + v_B \cdot D'] \cdot [A_A - w_A] - v_A \cdot D' \cdot [A_B - w_B]}{\Omega_1} \\
q_B(w_A, w_B) &= \frac{2 \cdot [-P'_A + v_A \cdot D'] \cdot [A_B - w_B] - v_B \cdot D' \cdot [A_A - w_A]}{\Omega_1}
\end{align*}
\]

where $\Omega_1 = 2 \cdot [P'_A - v_A \cdot D'] \cdot 2 \cdot [P_B - v_B \cdot D'] - [v_A \cdot D'] \cdot [v_B \cdot D'] > 0$ (A.3)

And deriving with respect to the input prices, we get:

\[
\begin{align*}
\frac{\partial q_A}{\partial w_A} &= \frac{2 \cdot [P'_B - v_B \cdot D']}{\Omega_1} < 0 \tag{A.4} \\
\frac{\partial q_A}{\partial w_B} &= \frac{v_A \cdot D'}{\Omega_1} > 0 \tag{A.5} \\
\frac{\partial q_B}{\partial w_A} &= \frac{v_B \cdot D'}{\Omega_1} > 0 \tag{A.6} \\
\frac{\partial q_B}{\partial w_B} &= \frac{2 \cdot [P'_A - v_A \cdot D']}{\Omega_1} < 0 \tag{A.7}
\end{align*}
\]

**Input prices**

Solving the first-order conditions for the input supplier under price discrimination, $\partial \Pi^{PD}/\partial w_A$ and $\partial \Pi^{PD}/\partial w_B$, we get:

\[
\begin{align*}
w_A &= \frac{[P'_A + v_A \cdot D'] \left[8 \cdot A_A \left[-P'_B + v_B \cdot D'\right] - 2 \cdot A_B \cdot [v_A - v_B] \cdot D'\right] - A_A \cdot v_B \cdot [v_A + v_B] D'^2}{16 \cdot [-P'_A + v_A \cdot D'] \cdot [-P'_B + v_B \cdot D'] - [v_A \cdot D' + v_B \cdot D']^2} \\
w_B &= \frac{[P'_B + v_B \cdot D'] \left[8 \cdot A_B \left[-P'_A + v_A \cdot D'\right] + 2 \cdot A_A \cdot [v_A - v_B] \cdot D'\right] - A_B \cdot v_A \cdot [v_A + v_B] D'^2}{16 \cdot [-P'_A + v_A \cdot D'] \cdot [-P'_B + v_B \cdot D'] - [v_A \cdot D' + v_B \cdot D']^2}
\end{align*}
\]

(A.8)

Solving $\partial \Pi^U/\partial w$, we obtain:

\[
w = \frac{A_A \cdot [-2 \cdot P'_B + v_B \cdot D'] + A_B \cdot [-2 \cdot P'_A + v_A \cdot D']}{4 \cdot [-P'_A - P'_B] + 2 \cdot [v_A + v_B] \cdot D'}
\]

(A.10)
Proof of Lemma 1

Using Eqs. (A.8) and (A.9), we get that \( w_B - w_A \) equals:

\[
\begin{align*}
A_B \cdot \left[ 8 \cdot P_A' \cdot P_B' + 5 \cdot v_A \cdot v_B \cdot D'^2 + v_A^2 \cdot D^2 + 2 \cdot v_A \cdot D' \cdot \left[ -4P_B' - P_A' \right] - 6 \cdot P_A' \cdot v_B \cdot D' \right] \\
16 \cdot \left[ -P_A' + v_A \cdot D' \right] \cdot \left[ -P_B' + v_B \cdot D' \right] - \left[ v_A \cdot D' + v_B \cdot D' \right]^2 \\
A_A \cdot \left[ 8 \cdot P_A' \cdot P_B' + 5 \cdot v_A \cdot v_B \cdot D'^2 + v_B^2 \cdot D^2 + 2 \cdot v_B \cdot D' \cdot \left[ -4P_A' - P_B' \right] - 6 \cdot P_B' \cdot v_A \cdot D' \right] \\
16 \cdot \left[ -P_A' + v_A \cdot D' \right] \cdot \left[ -P_B' + v_B \cdot D' \right] - \left[ v_A \cdot D' + v_B \cdot D' \right]^2
\end{align*}
\]

(A.11)

where the denominator is positive by the second-order conditions of the supplier maximization problem \((v_B/v_A > 7 - 4\sqrt{3} \approx 0.0718)\) is sufficient). Therefore, the condition in Lemma 1 follows straightforwardly as the terms multiplying \( A_A \) and \( A_B \) in Eq. (A.11) are positive.

Effect of price discrimination on output and prices

To simplify notation, we omit the arguments of the functions and let \( \tau \) be the input price in market \( A \) and \( \tau + t \) the charge in market \( B \). For a given \( t \in [0, t^*] \), the first-order condition of the supplier’s maximization profit is:

\[
\frac{\partial \Pi}{\partial \tau} = [q_A + q_B] + \tau \cdot \left[ \frac{\partial q_A}{\partial \tau} + \frac{\partial q_B}{\partial \tau} \right] + t \cdot \frac{\partial q_B}{\partial \tau}.
\]

(A.12)

This first-order condition defines implicitly \( \tau \) as a function of \( t \) in the following way:

\[
\frac{d\tau}{dt} = -\frac{\partial^2 \Pi / \partial \tau^2}{\partial^2 \Pi / \partial \tau^2} = -\frac{\left[ \frac{\partial q_A}{\partial w_A} + \frac{\partial q_A}{\partial w_B} + \frac{\partial q_B}{\partial w_A} + \frac{\partial q_B}{\partial w_B} \right]}{2 \cdot \left[ 4 \cdot P_A' - [3 \cdot v_A - v_B] \cdot D' \right] / \Omega_1}
\]

(A.13)

The marginal output effect is given by:

\[
\frac{dQ}{dt} = \frac{d\tau}{dt} \cdot \left[ \frac{\partial q_A}{\partial w_A} + \frac{\partial q_A}{\partial w_B} + \frac{\partial q_B}{\partial w_A} + \frac{\partial q_B}{\partial w_B} \right] + \left[ \frac{\partial q_A}{\partial w_B} + \frac{\partial q_B}{\partial w_B} \right]
\]

(A.14)

which can be simplified using Eq. (A.13) to:

\[
\frac{dQ}{dt} = \frac{1}{2} \cdot \left[ \frac{\partial q_A}{\partial w_B} - \frac{\partial q_B}{\partial w_A} \right] = \frac{[v_A - v_B] \cdot D'}{2 \cdot \Omega_1}
\]

(A.15)

where the last equality uses Eqs. (A.4)–(A.7).

The marginal effect on input prices follows from Eq. (A.14):

\[
\frac{dw_A}{dt} = \frac{d\tau}{dt} = \frac{4 \cdot P_A' - [3 \cdot v_A - v_B] \cdot D'}{\Omega_2}
\]

(A.17)

\[
\frac{dw_B}{dt} = \frac{d\tau}{dt} + 1 = \frac{3 \cdot v_B - v_A \cdot D' - 4 \cdot P_B'}{\Omega_2}
\]

(A.18)

where \( \Omega_2 = -4 \cdot [P_A' - v_A \cdot D'] - 4 \cdot [P_B' - v_B \cdot D'] + 2 \cdot [v_A + v_B] \cdot D' > 0 \)

(A.19)

From which Proposition 1 follows directly.
Welfare analysis when price discrimination changes quantities in the same direction

Consider that price discrimination changes both prices and both quantities in the same direction (either fall or rise). When the input price in both markets is higher than socially optimal under both pricing regimes, the bracketed terms of Eq. (15) are always positive and a sufficient condition for welfare improvement is that output increases in both markets. If the quantity decreases in both markets, price discrimination deteriorates welfare. Under price discrimination the input prices are higher than the welfare maximizing prices (see Eqs. (6) and (7)), but this is not necessarily true for the uniform price. Therefore the sufficient conditions for welfare improvement or deterioration in this case involve quantity changes (the sign of \(\frac{dq_A}{dt}\) and \(\frac{dq_B}{dt}\)) but also conditions so that output is contracted in both markets under uniform pricing. The following proposition summarizes the sufficient conditions that characterize the welfare change in this case.

**Proposition 7.** When:

(a) Time valuations are sufficiently different in that \(v_A - 5 \cdot v_B > -4 \cdot P_B' / D'\) and

(b) Congestion effects are not too high in that \(v_A \cdot D' < -5 \cdot P_B' + \sqrt{[5 \cdot P_B']^2 + 8 \cdot P_A' \cdot P_B'}\)

or the time valuations are not too different in that \(\frac{v_B}{v_A} > \frac{9 - \sqrt{73}}{4} \approx 0.114\),

Price discrimination changes the quantity in both markets in the same direction and:

(i) Increases welfare if \(\frac{A_B}{A_A} > \lambda_1\) (\(w_B > w_A\) holds and both quantities increase).

(ii) Decreases welfare if \(\lambda_0 < \frac{A_B}{A_A} < \lambda_1\) (\(w_B < w_A\) holds and both quantities decrease), where \(\lambda_0\) is defined in Appendix A.

Proof: see below.

The conditions (a) and (b) imply that price discrimination changes both input prices and output in both markets in the same direction. That is, either both input prices fall and both quantities rise or vice versa. Proposition 7 (i) is intuitive. When the congestion effects and time valuations are such that conditions (a) and (b) in Proposition 7 hold, \(A_B/A_A > \lambda_1\) implies that both prices fall and quantities rise. This, naturally, increases welfare because the input prices in both markets are higher than socially optimal in this case. Under these conditions, also the consumer surplus in each market increases. As discussed above, this can only occur when the ratio of reservation prices \(A_B/A_A\) is higher than the ratio of time valuations \(v_B/v_A\), which for conditions (a) and (b) to hold must be lower than \(1/5\). As a reference, \(\lambda_1\) is greater than \(1/2\) when \(v_B/v_A = 1/5\), so that the asymmetry of demand intercepts has to be significantly lower than the asymmetry of time valuations for welfare to increase with price discrimination.

In case (ii) of Proposition 7 the input prices rise and output falls in both markets, so that price discrimination necessarily decreases welfare if the prices were above socially optimal, which occurs when \(\lambda_0 < A_B/A_A\) (see Appendix A for the definition of \(\lambda_0\)). As in this case price discrimination increases the input prices and the differentiated prices are always higher than optimal (see Eqs. (6) and (7)), the uniform price is not necessarily higher than the socially optimal price of each market. For example, when \(q_B\) is relatively low and \(q_A\)
is relatively high, the differentiated input price set by the facility in market B is similar to the socially optimal price. Therefore, as the uniform price is lower than the differentiated prices in this case, w can be lower than the socially optimal price for market B. This is likely to happen when $A_B/A_A$ is sufficiently small, which explains why a lower bound for this ratio is needed for the sufficient condition ($\lambda_0 < A_B/A_A$). Again, as a reference, when $v_B/v_A = 1/5$, $\lambda_0 < 1/2$, so $A_B/A_A = 1/2$ is already small enough for the condition in Proposition 7 (ii) to hold.

**Proof of Proposition 7**

The first step of the proof is to show that $\tau - q_A \cdot P'_A - q_B \cdot v_B \cdot D'$ and $\tau + t - q_B \cdot P'_B - q_A \cdot v_A \cdot D'$ are always positive. From Eqs. (6) and (7) it follows that they are positive under price discrimination, so we need to show that they are positive for any value of $t$ between zero and the optimal difference between input prices. As all functions are linear, the sign of the derivative of the terms does not change (see above that all derivatives are constant) and it is enough to show that the terms are positive under uniform pricing (at $t = 0$). For $\tau - q_A \cdot P'_A - q_B \cdot v_B \cdot D'$, we show that the term decreases with $t$, which implies that when $A_B/A_A > \lambda_1$ holds the price discriminatory price is approached by increasing $t$ and therefore $\tau - q_A \cdot P'_A - q_B \cdot v_B \cdot D'$ has to be positive at any $t < t^*$:

$$
\frac{d}{dt} [\tau - q_A P'_A - q_B v_B D'] = \frac{8P_A^2 [ - P'_B + v_B D' ] + [v_A - v_B] [12P_B v_A + 11v_A - v_B] v_B D'] D'^2}{2 [2P'_A + 2P'_B - v_A - v_B] [4[-P'_A + v_A][-P'_B + v_B] - v_A v_B]}
+ \frac{P'_A [v_A D'^2 + 2P'_B [9v_A - 5v_B] D' + 17v_A v_B D'^2 - 12v_B^2]}{2 [2P'_A + 2P'_B - v_A - v_B] [4[-P'_A + v_A][-P'_B + v_B] - v_A v_B] < 0}
$$

(A.20)

In the case where $A_B/A_A < \lambda_1$, the discriminatory price is approached by making $t$ negative, so we need to assess directly $w - q_A \cdot P'_A - q_B \cdot v_B \cdot D'$. Substituting the values of $w, q_A(w)$ and $q_B(w)$, we obtain the following condition:

$$w - q_A \cdot P'_A - q_B \cdot v_B \cdot D' > 0 \Leftrightarrow \frac{A_B}{A_A} \cdot F_B > -F_A$$

(A.21)

where $F_A = -2P'_B [ - [3v_A + 11v_B] P'_A D' + v_B [7v_A + v_B] D'^2 + 4P_A^2 ] - 4P_B^2 [3P_A - 2v_A D'] + v_B D' [ - [5v_A + 12v_B] D' P'_A + v_B [7v_A + v_B] D'^2 + 8P_A^2 ] > 0$

$F_B = P_A [ - D' [v_A D' + 2[3v_A - 4v_B] P'_B + 6v_B^2 D'] ] + 2P_A^2 [v_A D' + 2P'_B - v_A D' [v_B - 5v_B] - 4[v_A - 2v_B] D' P'_B ]$

If $F_B > 0$, the condition in Eq. (A.21) always holds. If $F_B < 0$, then the condition is equivalent to $\frac{A_B}{A_A} < -F_B/F_B$, and as $-F_A/F_B > \lambda_1$ holds, $A_B/A_A < \lambda_1$ is sufficient. Therefore, $\tau - q_A \cdot P'_A - q_B \cdot v_B \cdot D'$ is positive for any value of $t$.

For $\tau + t - q_B \cdot P'_B - q_A \cdot v_A \cdot D'$ to be positive under uniform pricing, we assess its sign directly. Replacing $w, q_A(w)$ and $q_B(w)$, we obtain that it is positive when $A_B/A_A > \lambda_0$, 31
where:

\[
\lambda_0 = \left[ P_B' \left[ -6v_A^2D^2 + 2[4v_A - 3v_B]P_A' D' - v_B^2D^2 \right] + 2P_B^2[2P_A' + v_BD'] + v_B \left[ 4[v_B - 2v_A]P_A' + v_A[5v_A - v_B]D' \right] D^2 \right] \\
\cdot 2P_A' \left[ (11v_A + 3v_B)P_B' D' - v_A[v_A + 7v_B]D^2 - 4P_B^2 \right] - 4P_A^2[2v_BD' - 3P_B'] \\
- v_A \left[ (12v_A + 5v_B)P_B' D' - v_A[v_A + 7v_B]D^2 + 8P_B^2 \right]^{-1}
\]

(A.22)

and, as \( \lambda_0 < \lambda_1 \), \( A_B/A_A > \lambda_1 \) is a sufficient condition for \( \tau + t - q_B \cdot P_B' - q_A \cdot v_A \cdot D' \) to be positive. In the case where \( A_B/A_A < \lambda_1 \), \( A_B/A_A > \lambda_0 \) is also needed.

The second part of the proof is to show that both quantities move in the same direction.

The marginal effect on downstream firm’s quantities are:

\[
\frac{dq_A}{dt} = \frac{\partial q_A}{\partial v_A} \cdot \frac{d\tau}{dt} + \frac{\partial q_A}{\partial v_B} \cdot \left[ \frac{d\tau}{dt} + 1 \right] \\
= 2 \cdot \left[ P_B' - v_B \cdot D' \right] \cdot 4 \cdot P_A' - [5 \cdot v_A - v_B] \cdot D' \right] - v_A \cdot [v_A + v_B] \cdot D^2 \\
\Omega_2
\]

\[
\frac{dq_B}{dt} = \frac{\partial q_B}{\partial v_A} \cdot \frac{d\tau}{dt} + \frac{\partial q_B}{\partial v_B} \cdot \left[ \frac{d\tau}{dt} + 1 \right] \\
= -2 \cdot \left[ P_A' - v_A \cdot D' \right] \cdot 4 \cdot P_B' - [5 \cdot v_B - v_A] \cdot D' \right] + v_B \cdot [v_A + v_B] \cdot D^2 \\
\Omega_2
\]

(A.23)

(A.24)

From Eq. A.24 it follows that part (a) of Proposition 7, \( v_A - 5 \cdot v_B > -4 \cdot P_B'/D' \), is sufficient for \( dq_B/dt > 0 \) as all other terms are positive.

As \( \Omega_2 > 0 \), we focus on the numerator of (A.23) to determine the sign of \( dq_A/dt \). Denote \( I \) the numerator and let \( v_B = \phi \cdot v_A \) where \( \phi \) is a constant in [0, 1]. This allows for focusing on a sufficient condition for \( v_A \) by taking into account that \( v_B < v_A \) must always hold. Solving \( I = 0 \) for \( v_A \), we get the following roots:

\[
r_1 = \frac{-P_B' \cdot [5 - \phi] - 4 \cdot P_A' \cdot \phi + \sqrt{[-P_B' \cdot [5 - \phi] - 4 \cdot P_A' \cdot \phi]^2 + 8 \cdot P_A' \cdot P_B' \cdot [1 - 9 \cdot \phi + 2 \cdot \phi^2]}}{D' \cdot [1 - 9 \cdot \phi + 2 \cdot \phi^2]}
\]

(A.25)

\[
r_2 = \frac{-P_B' \cdot [5 - \phi] - 4 \cdot P_A' \cdot \phi - \sqrt{[-P_B' \cdot [5 - \phi] - 4 \cdot P_A' \cdot \phi]^2 + 8 \cdot P_A' \cdot P_B' \cdot [1 - 9 \cdot \phi + 2 \cdot \phi^2]}}{D' \cdot [1 - 9 \cdot \phi + 2 \cdot \phi^2]}
\]

(A.26)

To prove that the condition in part (b) of Proposition 7 is sufficient for \( dq_A/dt > 0 \), we distinguish two cases. First, when \( [1 - 9 \cdot \phi - 2 \cdot \phi^2] > 0 \), which is equivalent to \( \phi < [9 - \sqrt{73}]/4 \approx 0.114 \), \( r_2 \) is negative, \( r_1 \) is positive, and \( \partial^2 I/\partial v_A^2 < 0 \). Therefore for all values of \( v_A \) in \([0, r_1] \), \( dq_A/dt > 0 \). The minimum value of \( r_1 \) when \( [1 - 9 \cdot \phi + 2 \cdot \phi^2] > 0 \) is achieved at \( \phi = 0 \), so that a sufficient condition is:

\[
v_A < r_1 |_{\phi=0} = \frac{-P_B' \cdot 5 - \sqrt{[-P_B' \cdot 5]^2 + 8 \cdot P_A' \cdot P_B'}}{D'}
\]

(A.27)
which is the condition in part (b) of Proposition 7. In the case where \([1 - 9 \cdot \phi + 2 \cdot \phi^2] < 0\), which is equivalent to \(\phi > [9 - \sqrt{73}]/4 \approx 0, 114\), both roots are negative and \(\partial^2 I/\partial w_A^2 > 0\) so that for all positive values of \(v_A, dq_A/dt > 0\) holds. This completes the proof that the condition \(v_A < r_1 |_{\phi=0} \) or \(v_B/v_A > [9 - \sqrt{73}]/4 \approx 0, 114\) is sufficient for \(dq_A/dt > 0\) to hold.

Proof of Proposition 2

- Welfare improvement: \(\frac{A_B}{A_A} > \lambda_1\)

When \(v_A - 3 \cdot v_B < -4 \cdot P'_B/D'\) holds, it follows from Eqs. (A.17) and (A.18) that \(d\tau/dt < 0\) and \(d\tau/dt + 1 > 0\). This implies that \(dq_A/dt > 0\) and \(dq_B/dt < 0\) (see Eqs. (A.23) and (A.24)). From Eq. (15), it follows then that showing that \(w_A - q_A \cdot P'_A - q_B \cdot v_B \cdot D'\), which is positive (see Proof of Proposition 7), is greater than \(w_A - t - q_B \cdot P'_B - q_A \cdot v_A \cdot D'\) for any value of \(t \in [0, t^*]\) is sufficient for \(dW/dt > 0\). Denote \(f(t)\) the difference between these two terms; we prove that the condition in Proposition 2 is sufficient for \(f(t) > 0\) to hold. Formally,

\[
\begin{align*}
 f(t) &= -q_A \cdot \left[P'_A - v_A \cdot D'\right] + q_B \cdot \left[P'_B - v_B \cdot D'\right] - t \\
 \frac{df}{dt} &= - \frac{dq_A}{dt} \cdot \left[P'_A - v_A \cdot D'\right] + \frac{dq_B}{dt} \cdot \left[P'_B - v_B \cdot D'\right] - 1 \\
 \frac{df}{dt} &= -8 \left[P'_A + P'_B\right] - [P'_B + v_A \cdot D' + P'_A - P'_B - v_B \cdot D'] \\
 & \left\{ - [v_A - v_B]^2 \left[v_A + v_B \right] \cdot D'' + 5v_A + v_B \left[-P'_B \cdot v_B - P'_A \cdot v_A\right] \right\} \\
 \frac{df}{dt} &= \frac{1}{\Omega_2} \\
\end{align*}
\]

As \(df/dt < 0\), \(f(t^*) > 0\) is sufficient for \(dW/dt > 0\). Using that \(t^* = w_B - w_A\) and Eq. (A.11), we get:

\[
\begin{align*}
 f(t^*) &= \frac{A_A \cdot \left[12 \cdot P'_A \cdot P'_B + 10 \cdot v_A \cdot v_B + 2 \cdot v_A^2 + 3 \cdot v_B \cdot \left[-4P'_A - P'_B\right] - 11 \cdot P'_B \cdot v_A\right]}{16 \cdot \left[P'_A + v_a \cdot D'\right] \cdot \left[-P'_B + v_B \cdot D'\right] - [v_A \cdot D' + v_B \cdot D']^2} \\
 & \frac{- \frac{A_B \cdot \left[12 \cdot P'_A \cdot P'_B + 10 \cdot v_A \cdot v_B + 2 \cdot v_A^2 + 3 \cdot v_B \cdot \left[-4P'_B - P'_A\right] - 11 \cdot P'_A \cdot v_B\right]}{16 \cdot \left[-P'_A + v_A \cdot D'\right] \cdot \left[-P'_B + v_B \cdot D'\right] - [v_A \cdot D' + v_B \cdot D']^2}} \\
\end{align*}
\]

from which the result in Proposition 2 follows directly, as the denominator is positive by the second-order conditions and the terms multiplying \(A_A\) and \(A_B\) in Eq. (A.32) are positive.

Finally, to show that the interval \([\lambda_1, \lambda_2]\) is non-empty when \(\frac{P'_A}{P'_B} < \frac{v_A}{v_B}\) holds, we look at \(\lambda_2 - \lambda_1\):

\[
\begin{align*}
 \lambda_2 - \lambda_1 &= \frac{L_1 \cdot \left[-P'_B v_A + P'_A v_B\right]}{L_2 \cdot L_3} \\
\end{align*}
\]

where \(L_1 = 16 \cdot \left[-P'_A + v_A \cdot D'\right] \cdot \left[-P'_B + v_B \cdot D'\right] - [v_A \cdot D' + v_B \cdot D']^2 > 0\)

\(L_2 = 8 \cdot P'_A \cdot P'_B + 5 \cdot v_A \cdot v_B + v_A^2 + 2 \cdot v_A \cdot \left[-4P'_B - P'_A\right] - 6 \cdot P'_A \cdot v_B > 0\)

\(L_3 = 12 \cdot P'_A \cdot P'_B + 10 \cdot v_A \cdot v_B + 2 \cdot v_A^2 + 3 \cdot v_A \cdot \left[-4P'_B - P'_A\right] - 11 \cdot P'_A \cdot v_B > 0\)

where \(L_1 > 0\) follows from \(\partial^2 \Pi/\partial w_A^2 \cdot \partial^2 \Pi/\partial w_B^2 > [\partial^2 \Pi/\partial w_A \partial w_B]^2\), a second-order condition that we assume to hold. Therefore, \(\lambda_2 - \lambda_1 \geq 0 \iff -P'_B v_A + P'_A v_B > 0\), which proves the result.
- Welfare deterioration: $\frac{\Delta q}{\Delta A} < \lambda_1$

As the price discriminating behavior is approached by making $t$ negative in this case, the effect of price discrimination on welfare, output and prices have the opposite sign than the marginal effect. That is, as $w_A > w_B$ holds, welfare decreases when $dW/dt > 0$. As $v_A - 3 \cdot v_B < -4 \cdot P_B' / D'$ holds, $dA/dt > 0$ and $dA'/dt < 0$. Therefore, again, $w_A - q_A \cdot P_B' - q_B \cdot v_B \cdot D' > w_A + t - q_B \cdot P_B' - q_A \cdot v_A \cdot D'$ for any value of $t \in [-t^*, 0]$ is sufficient for $dW/dt > 0$ and thus for welfare deterioration. As $df/dt < 0$, the sufficient condition in this case is that $f(0) > 0$. Using Eqs. (A.1) and (A.2):

$$f(0) = \frac{A_A \left[ -P_B' + v_B D' \right] \left[ -4P_A' + v_B D' + 5v_A D' \right] - A_B \left[ -P_A' + v_A D' \right] \left[ -4P_B' + v_A D' + 5v_B D' \right]}{L_1}$$

(A.34)

from which the condition $\frac{\Delta q}{\Delta A} < \lambda_3$ follows straightforwardly.

Appendix B. Calculations and proofs for Section 4

Input prices

Solving the first-order conditions for the input supplier under price discrimination, $\partial W_D/\partial w_A$ and $\partial W_D/\partial w_B$, we get:

$$w_A = \frac{A_B \cdot \left[ -2 \cdot P_A' \cdot [v_A + 2 \cdot v_B] \cdot D' + 4 \cdot v_A \cdot v_B \cdot D'^2 \right]}{-P_A' + 2 \cdot v_A \cdot D'] \cdot [-7 \cdot P_B' + 8 \cdot v_B \cdot D'] - 2 \cdot [v_A \cdot D' + v_B \cdot D']^2}$$

$$+ A_B \cdot \left[ 2 \cdot v_B \cdot [v_A + v_B] \cdot D'^2 - P_A' \cdot [-7 \cdot P_B' + 8 \cdot v_B \cdot D'] \right]$$

$$- \frac{-P_A' + 2 \cdot v_A \cdot D'] \cdot [-7 \cdot P_B' + 8 \cdot v_B \cdot D'] - 2 \cdot [v_A \cdot D' + v_B \cdot D']^2}{-P_A' + 2 \cdot v_A \cdot D'] \cdot [-7 \cdot P_B' + 8 \cdot v_B \cdot D'] - 2 \cdot [v_A \cdot D' + v_B \cdot D']^2}$$

(A.1)

$$w_B = \frac{A_B \cdot \left[ -P_B' \cdot [4 \cdot v_A - 3 \cdot v_B] \cdot D' + 4 \cdot [v_A - v_B] \cdot v_B \cdot D'^2 \right]}{-P_A' + 2 \cdot v_A \cdot D'] \cdot [-7 \cdot P_B' + 8 \cdot v_B \cdot D'] - 2 \cdot [v_A \cdot D' + v_B \cdot D']^2}$$

$$+ \frac{A_B \cdot \left[ -P_B' \cdot [-3 \cdot P_B' + 4 \cdot v_B \cdot D'] + v_A \cdot D' \cdot [-6 \cdot P_B' + 6 \cdot v_B - 2 \cdot v_A] \cdot D'] \right]}{-P_A' + 2 \cdot v_A \cdot D'] \cdot [-7 \cdot P_B' + 8 \cdot v_B \cdot D'] - 2 \cdot [v_A \cdot D' + v_B \cdot D']^2}$$

(B.2)

Solving $\partial W_D/\partial w$, we obtain:

$$w = \left[ A_A \cdot \left[ 2(-P_B' + v_B D') \cdot [v_A D' \cdot 4v_A D' - 5v_B D' - 6P_A] + P_A' \left[ -4P_B' + 7v_B D' \right] \right] \right.\left. - v_B^2 D'^2 \cdot [-P_B' - P_A'] \right]$$

$$+ 2A_B \cdot \left[ - P_A' \cdot [6P_B' + 8v_B D'] \cdot [-P_A' + 2v_A D'] - P_B' \cdot [2v_A + v_B] D' \right.$$

$$\left. + v_A D' \cdot [-P_B' \cdot [6v_A + v_B] D' + v_A D'^2 \cdot 7v_B - v_A] + 3P_B' v_A D'] \right] / \Omega_3$$

where $\Omega_3 = 4P_A' P_B' - 7P_A' - 2P_B' + 16v_A D' \right] + v_B D' \left[ -16P_B^2 + 6P_A v_A D' - 28P_B v_A D' - 4v_A^2 D'^2 \right.$

$$\left. + v_B D' \left[ -4P_A' - v_B D' \left[ -8P_A' - P_B' + 16v_A D'] + 4v_A D' \cdot 7v_A D' + 3v_B D' - 3P_B' \right] \right]$$

(B.3)
Effect of price discrimination on output and prices

To simplify notation, we again omit the arguments of the functions and let \( \tau \) be the input price in market \( A \) and \( \tau + t \) the charge in market \( B \). The marginal effect on \( \tau \) is:

\[
\frac{d\tau}{dt} = -\frac{\partial^2 W_D/\partial \tau \partial t}{\partial^2 W_D/\partial \tau^2}, \tag{B.4}
\]

where,

\[
\frac{\partial^2 W_D}{\partial \tau^2} = \left[ \frac{\partial q_A}{\partial w_A} + \frac{\partial q_A}{\partial w_B} \right] \left[ \frac{\partial q_A}{\partial w_A} + \frac{\partial q_A}{\partial w_B} \right] \left[ P_A' - v_A D' \right] - v_A D' \left[ \frac{\partial Q}{\partial w_A} + \frac{\partial Q}{\partial w_B} \right] + \frac{\partial q_B}{\partial w_A} + \frac{\partial q_B}{\partial w_B} \tag{B.5}
\]

\[
\frac{\partial^2 W_D}{\partial \tau \partial t} = \left[ \frac{\partial q_A}{\partial w_A} + \frac{\partial q_A}{\partial w_B} \right] \left[ \frac{\partial q_A}{\partial w_A} + \frac{\partial q_A}{\partial w_B} \right] \left[ P_A' - v_A D' \right] - v_A D' \left[ \frac{\partial Q}{\partial w_A} + \frac{\partial Q}{\partial w_B} \right] + \frac{\partial q_B}{\partial w_A} + \frac{\partial q_B}{\partial w_B} \tag{B.6}
\]

Substituting Eqs. (A.4)–(A.7) in Eq. (B.4) yields:

\[
\frac{d\tau}{dt} = \frac{4P_A^2[-7P_B + 8v_B D'] - P_A' [6v_A^2 - 60v_A v_B + 8v_B^2]D'^2 - 6P_B' [v_B - 10v_A]D']}{\Omega_3} + 2v_A D' \left[ -P_B' + v_B D' \right] \left[ 3v_B - 14v_A \right] D'^2 + v_A [2v_A + v_B] D' \left[ 35 - 2B' \right] \tag{B.7}
\]

where \( \Omega_3 \) is positive as we assume that the second-order conditions of the supplier’s maximization problem under uniform pricing holds (i.e. \( \partial^2 W_D/\partial w^2 < 0 \)). This, together with \( v_A > v_B \) and \( \partial^2 W_D/\partial w_A^2 \cdot \partial^2 W_D/\partial w_B^2 > [\partial^2 W_D/\partial w_A \partial w_B]^2 \), which again holds as we assume that the second-order conditions of the supplier’s maximization problem under price discrimination holds, imply that \( d\tau/dt < 0 \). As \( dw_A/dt = d\tau/dt \), we get that price discrimination decreases the input price in market \( A \).

The marginal effect on \( w_B \) is:

\[
\frac{dw_B}{dt} = \frac{d\tau}{dt} + 1 = -\frac{4P_B^2 [2P_A P_B' - v_A D' [4P_B' + P_A']]}{\Omega_3} + v_B D' \left[ -P_A' [-10P_B + 4v_A D'] - P_B' [18v_A - v_B] D' + 2v_A [v_A + v_B] D'^2 \right] \tag{B.8}
\]

which proves that price discrimination increases the input price in market \( B \).

Using Eqs. (A.4)–(A.7), (A.15) and (B.7), we get:

\[
\frac{dQ}{dt} = \frac{2 [-P_B' + v_B D'] [-P_A' + [v_A - v_B] D'] - 2 [P_A' - v_A D'] [-4P_B' + 3v_B - v_A] D'] + P_B' v_B D'}{\Omega_3} \tag{B.9}
\]

From Eq. (B.9) it follows that \( v_A - 3 \cdot v_B < -4P_B'/D' \) is sufficient for \( dQ/dt > 0 \) to hold.
Proof of Proposition 4

Subtracting the value of the total welfare in Eq. (3) when evaluated at \(\{w_A, w_B\}\) and at \(w\), we obtain:

\[
W(w_A, w_B) - W(w) = \frac{[w_B - w_A]}{2 \cdot \Omega_4} \cdot \frac{[A_A \cdot \lambda_4^N - A_B \lambda_4^D]}{\Omega_4^4} \tag{B.10}
\]

where \(\Omega_4 = 8P_B[4P_A' + P_B'][-P_A' + 2v_AD'] + 4P_A'^2P_B' + 2v_A^2D^2[3P_A' - 14P_B' - 2v_AD'] - v_BD' \left[4v_AD'[12P_A' + 3P_B' - 7v_AD'] - [4P_A' + v_BD'][8P_A' + P_B' - 4v_AD'] \right] \]

As \(w_B > w_A\) holds, the sign of the welfare change is given by \([A_A \cdot \lambda_4^N - A_B \lambda_4^D]\), where \(\lambda_4^N\) and \(\lambda_4^D\) are given by Eqs. (B.11) and (B.12) respectively. As \(\lambda_4^N\) is the numerator of \(\lambda_4\) and \(\lambda_4^D\) is the denominator of \(\lambda_4\) and it is positive, the result proves Proposition 4.

\[
\lambda_4^N = \left[4v_A^6 + 7v_B^6 - 91v_A^2v_B^4 + 231v_A^4v_B^2 + 33v_A^3v_B^3 + 1350v_A^2v_B - 198v_A^5v_B \right] D^6
\]

\[
+ \left[-3v_A^5 - 355v_Bv_A^3 + 6070v_B^2v_A^2 - 164v_B^3v_A + 116v_B^4 \right] P_A^2 D^4
\]

\[
+ 2v_B \left[35v_A^2 - 155v_Bv_A + 32v_B^2 \right] P_A^2 D^3 + 87v_B^2D^2 P_A^4
\]

\[
+ \left[-4v_A^4 + 509v_Bv_A^3 - 4890v_B^2v_A^2 + 71v_B^3v_A - 320v_B^4 \right] P_A^4 D^3 \tag{B.11}
\]

\[
- \left[108P_A' - 160v_A^3 + 25v_B^3 - 186v_Av_B^2 - 280v_A^2v_B \right] D^3
\]

\[
+ \left[488v_A^2 + 288v_Bv_A + 139v_B^2 \right] P_A'^2 D^2 - 4[99v_A + 10v_B] P_A'^2 D'
\]

\[
- 4 \left[3P_A' - 10v_AD' - 7v_BD' \right] P_A'^4
\]

\[
+ \left[6v_A^5D^6 - 3 \left[3P_A' + 40v_BD' \right] v_A^5D^5 + v_B \left[325P_A' + 538v_BD' \right] v_A^4D^5 \right. \\

\left. - v_B \left[1947v_BD'P_A'^2 + 254P_A'^2 + 124v_B^2D^2 \right] v_A^3D^4 \right]
\]

\[
+ v_B \left[331v_B^2D^2P_A'^2 + 2490v_BD'P_A^2 + 56P_A'^2 + 54v_B^3D^3 \right] v_A^2D^3
\]

\[
- 4v_B^2 \left[4P_A' + v_BD' \right] \left[-4v_BD'P_A' + 84P_A'^2 + 5v_B^2D^2 \right] v_AD^3
\]

\[
+ 2v_B^2 \left[4P_A' + v_BD' \right]^2 \left[-2v_BD'P_A' + 8P_A'^2 + v_B^2D^2 \right] v_BD^3 \tag{B.12}
\]

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\[
\lambda_D^4 = \left[ v_A \left[ 28 v_A^5 - 80 v_B v_A^4 - 1075 v_B^2 v_A^3 - 247 v_B^4 v_A + 155 v_B^5 v_A - 13 v_B^6 \right] \\
+ \left[ 110 v_A^4 - 1187 v_B v_A^3 - 5083 v_B^2 v_A^2 + 160 v_B^3 v_A + 36 v_B^4 \right] P_A D'^4 \\
- 5 \left[ 7 v_A^3 - 174 v_B^2 v_A^2 - 536 v_B^3 v_A + 16 v_B^4 \right] P_A^2 D'^3 - 16 v_B [13 v_A + 32 v_B] P_A D'^2 \\
+ \left[ -100 v_A^5 + 576 v_B v_A^4 + 4049 v_B^2 v_A^3 + 104 v_B^3 v_A^2 - 155 v_B^4 v_A + 6 v_B^5 \right] P_A D'^2 \right] P_B D^6 \\
+ \left[ -24 v_A^4 - 5 v_B^4 + 123 v_A^3 v_B^2 - 482 v_A^2 v_B^3 - 685 v_A v_B^4 \right] D'^4 \\
- \left[ 223 v_A^3 + 1248 v_B v_A^2 + 152 v_B^3 \right] P_A^2 D'^2 + 4 [21 v_A + 80 v_B] P_A^3 D'^3 \\
+ \left[ 144 v_A^2 + 1478 v_B v_A + 543 v_B^2 v_A - 72 v_B^3 \right] P_A D'^3 \right] P_B^2 \left[ P_A - 2 v_B D' \right] \\
- \left[ 68 v_A^2 D' - 4 \left[ 42 P_A - 53 v_B D' \right] v_A D' + \left[ 2 P_A - 5 v_B D' \right] \left[ 6 P_A + v_B D' \right] \right] P_B \left[ P_A - 2 v_B D' \right]^2 \\
- 28 \left[ P_A - 2 v_A D' \right]^3 P_B^4 \\
+ \left[ 2 v_A^6 - 8 v_B v_A^5 - 94 v_B^2 v_A^4 + 412 v_B^3 v_A^3 + 34 v_B^4 v_A^2 - 28 v_B^5 v_A \right] D'^4 \\
+ \left[ -5 v_A^5 - 18 v_B^2 v_A^4 + 157 v_B^3 v_A^3 - 269 v_B^4 v_A^2 + 103 v_B^5 v_A \right] P_A D'^3 \\
+ \left[ 3 v_A^4 - 412 v_B v_A^3 + 2088 v_B^2 v_A^2 + 597 v_B^3 v_A^2 - 120 v_B^4 v_A \right] P_A^2 D'^2 \\
- \left[ 1200 v_B^3 v_A^2 + 480 v_A v_B^3 v_A + 42 v_B^2 v_B \right] P_A^3 D' + 128 v_B^3 v_B D'^3 \\
+ \left[ 16 v_B^4 D'^2 P_A^2 + 256 v_B^3 D' P_A^3 + 256 v_B^4 P_A^4 \right] v_B D'^3 \right] \\
\] (B.12)

**Appendix C. Calculations and proofs for Section 5**

The calculations and proofs are more brief in this section as they follow the same logic as the ones in the previous sections. Under downstream perfect price discrimination the derived demands are:

\[
q_A(w_A, w_B) = \frac{[-P_B' + 2 \cdot v_B \cdot D'] \cdot [A_A - w_A] - v_A \cdot D' \cdot [A_B - w_B]}{\Omega_1} \tag{C.1}
\]

\[
q_B(w_A, w_B) = \frac{[-P_A' + 2 \cdot v_A \cdot D'] \cdot [A_B - w_B] - v_B \cdot D' \cdot [A_A - w_A]}{\Omega_1} \tag{C.2}
\]

where \( \Omega_1' = [P_A' - 2 \cdot v_A \cdot D'] \cdot [P_B' - 2 \cdot v_B \cdot D'] - [v_A \cdot D'] \cdot [v_B \cdot D'] > 0 \) \tag{C.3}
**Private facility**

Solving the first-order conditions for the private input supplier under price discrimination we get:

\[
w_A = \frac{[-P'_A + 2 \cdot v_A \cdot D'] \left[2 \cdot A_A \cdot [-P'_B + 2 \cdot v_B \cdot D'] - A_B \cdot [v_A - v_B] \cdot D'\right] - A_A \cdot v_B \cdot [v_A + v_B]D'^2}{4 \cdot [-P'_A + 2 \cdot v_A \cdot D'] \cdot [-P'_B + 2 \cdot v_B \cdot D'] - [v_A \cdot D' + v_B \cdot D']^2}
\]

\[
w_B = \frac{[-P'_B + 2 \cdot v_B \cdot D'] \left[2 \cdot A_B \cdot [-P'_A + 2 \cdot v_A \cdot D'] + A_A \cdot [v_A - v_B] \cdot D'\right] - A_B \cdot v_A \cdot [v_A + v_B]D'^2}{4 \cdot [-P'_A + 2 \cdot v_A \cdot D'] \cdot [-P'_B + 2 \cdot v_B \cdot D'] - [v_A \cdot D' + v_B \cdot D']^2}
\]

\[
w = \frac{A_A \cdot [-P'_B + v_B \cdot D'] + A_B \cdot [-P'_A + v_A \cdot D']}{2 \cdot [-P'_A - P'_B] + 2 \cdot [v_A + v_B] \cdot D'}
\]

By subtracting both values we obtain that \(w_B - w_A > 0\) if and only if \(A_B/A_A > \lambda'_1\) where \(\lambda'_1\) is given by:

\[
\lambda'_1 = \frac{2 \cdot P'_A \cdot P'_B + v_A \cdot v_B \cdot D'^2 + v_B \cdot D'^2 + v_B \cdot D' \cdot [-4P'_A - P'_B] - 3 \cdot P'_B \cdot v_A \cdot D'}{2 \cdot P'_A \cdot P'_B + 5 \cdot v_A \cdot v_B \cdot D'^2 + v_B \cdot D'^2 + v_A \cdot v_B \cdot D' \cdot [-4P'_B - P'_A] - 3 \cdot P_A \cdot v_B \cdot D'}
\]

Using the same methodology as in Appendix A, we obtain:

\[
\frac{dQ}{dt} = \frac{[v_A - v_B] \cdot D'}{2 \cdot \Omega'_1} > 0,
\]

\[
\frac{dw_A}{dt} = \frac{2 \cdot P'_A \cdot [3 \cdot v_A - v_B] \cdot D'}{\Omega'_2} < 0,
\]

\[
\frac{dw_B}{dt} = \frac{[3 \cdot v_B - v_A] \cdot D' - 2 \cdot P'_B}{\Omega'_2},
\]

\[
\Omega'_2 = -2 \cdot \left[ P'_A + P'_B - v_A \cdot D' - v_B \cdot D' \right] > 0
\]

which proves that when \(v_B/v_A \geq 1/3\), the prices move in opposite directions. As a consequence, quantities also move in the opposite direction with price discrimination.

The marginal welfare effect under downstream perfect price discrimination is:

\[
\frac{dW}{dt} = \frac{dq_A}{dt} \cdot \left[w_A - [q_B \cdot v_B \cdot D']\right] + \frac{dq_B}{dt} \cdot \left[w_A + t - [q_A \cdot v_A \cdot D']\right],
\]

Also in this case it is straightforward to show that the bracketed terms are positive at the discriminating input prices and that \(w_A - [q_B \cdot v_B \cdot D']\) decreases with \(t\), so that it is positive for all values of \(t\). Moreover, just as in Appendix A the difference between the two terms, \(f'(t^*)\), decreases with \(t\). Then, when \(A_B/A_A > \lambda'_1\), \(f'(t^*) > 0\) is sufficient for \(dW/dt > 0\). Using the differentiated prices above, we obtain:

\[
f'(t^*) = \frac{A_A \cdot [-P'_B + 2v_B \cdot D'] \cdot [-2P'_A + 5v_A + v_B] \cdot D']}{4 \cdot [-P'_A + 2 \cdot v_A \cdot D'] \cdot [-P'_B + 2 \cdot v_B \cdot D'] - [v_A \cdot D' + v_B \cdot D']^2} - \frac{A_B \cdot [-P'_A + 2 \cdot v_A \cdot D'] \cdot [-2P'_B + 5v_B + v_A] \cdot D']}{4 \cdot [-P'_A + 2 \cdot v_a \cdot D'] \cdot [-P'_B + 2 \cdot v_B \cdot D'] - [v_A \cdot D' + v_B \cdot D']^2}
\]
from which it follows that welfare increases when \( \lambda'_1 < A_B/A_A < \lambda'_2 \), where
\[
\lambda'_2 = \frac{\left[ -P'_B + 2v_B \cdot D' \right] \cdot \left[ -2P'_A + 5v_A + v_B \cdot D' \right]}{\left[ -P'_A + 2v_A \cdot D' \right] \cdot \left[ -2P'_B + 5v_B + v_A \cdot D' \right]} \tag{C.14}
\]
In the case where \( \lambda'_1 > A_B/A_A \), \( f'(0) > 0 \) is sufficient for welfare to decrease with price discrimination by a private facility. Using the uniform price derived above we get:
\[
f'(0) = \frac{A_A \cdot L_A - A_B \cdot L_B}{\Omega'^2 \cdot [P'_A - 2 \cdot v_A \cdot D'] \cdot [P'_B - 2 \cdot v_B \cdot D'] - [v_A \cdot D'] \cdot [v_B \cdot D']}
\]
\[
L_A = \left[ P'_B \left[ 2P'_A + P'_B - 3v_AD' \right] v_AD' + \left[ 5v_AD' \right] P'_A \left[ -P'_B + 4v_AD' \right] - P'_A \left[ -P'_B + 4v_AD' \right] \right] v_B D'
- \left[ 3P'_A + P'_B - 6v_AD' \right] v_B^2 D'^2 + v_B^2 D^2
\]
\[
L_B = \left[ \left[ -P'_A \left[ -P'_B + v_AD' \right] + v_AD' \left[ -3P'_B + v_AD' \right] \right] v_AD'
+ \left[ -P'_A + 2v_AD' \right] \left[ -P'_A - 2P'_B + 3v_AD' \right] v_B D' + \left[ -3P'_A + 5v_AD' \right] v_B^2 D'^2 \right]
\]
from which it follows that welfare decreases when \( A_B/A_A < \min[\lambda'_1, \lambda'_3] \), where
\[
\lambda'_3 = \frac{L_A}{L_B} \tag{C.16}
\]

**Public facility**

Solving the first-order conditions for the public supplier, we obtain the following prices:

\[
w'_A = \frac{2 \cdot v_B \cdot \left[ A_B \cdot \left[ -P'_A + 2 \cdot v_A \cdot D' \right] - A_A \cdot \left[ v_A + v_B \right] \cdot D' \right]}{-P'_A + 2 \cdot v_A \cdot D' \cdot -3P'_B + 8 \cdot v_B \cdot D' - 2[v_A + v_B]^2 \cdot D'^2}
\]
\[
w'_B = \frac{A_A \cdot \left[ -P'_B - 2v_A \cdot D' + 4 \cdot \left[ v_A - v_B \right] \cdot v_B \cdot D' \right]}{-P'_A + 2 \cdot v_A \cdot D' \cdot -3P'_B + 8 \cdot v_B \cdot D' - 2[v_A + v_B]^2 \cdot D'^2}
+ \frac{A_B \cdot \left[ -P'_A - 4v_B \cdot D' + 2v_A \cdot D' \cdot -P'_B + 3 \cdot v_B - v_A \cdot D' \right]}{-P'_A + 2 \cdot v_A \cdot D' \cdot -3P'_B + 8 \cdot v_B \cdot D' - 2[v_A + v_B]^2 \cdot D'^2}
\]
\[
w' = \frac{A_A \cdot \left[ \left[ -P'_A + 2 \cdot v_A \cdot D' \right] \cdot \left[ -P'_B \left[ 2v_A - v_B \right] \cdot D' + 4v_B \left[ v_A - v_B \right] \cdot D'^2 \right] \right.}{-v_B D' \cdot \left[ -2P_B v_A - P'_B v_B + 2v_A v_B \cdot D' \right] \cdot D'}
+ \frac{A_B \cdot \left[ \left[ -P'_A + 2v_A D' \right] \left[ -P'_A + 2v_AD' \right] \left[ -P'_B + 4v_B D' \right] - v_A v_B D'^2 - 2v_B^3 D'^2 - P'B v_B D' \right]}{-v_A v_B D'^2 P'_A} / \Omega_3 \tag{C.19}
\]
where \( \Omega_3 = \left[ -P'_A + 2v_AD' \right] \left[ 3P'_A P'_B + 2P'^2_B - 6P'_B v_A D' - 2v_A^2 D'^2 \right] + \left[ P'_B - 4v_A D' \right] v_B^2 D'^2
+ \left[ -4P'^2_A + 3P'_A P'_B - 14P'_A v_A D' - 4P_B v_B D' + 14v_A^2 D'^2 \right] v_B D' \)

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Subtracting the value of the total welfare under discriminating and uniform prices, we obtain:

$$\Delta W = \frac{[w_B' - w'_A]}{\Omega_3} \cdot \frac{[A_A \cdot \lambda_4^{N'} - A_B \lambda_4^{D'}]}{4 \cdot \Omega_4^2}$$

(C.20)

$$\Omega_4' = -\left[6P_B'v_AD' - P'_A\left[3P_B - 8v_BD'\right] + 2 \left[v_A^2 - 6v_Av_B + v_B^2\right] \cdot D'^2\right]$$

As $w_B' > w'_A$ holds, it follows that the sign of the welfare change is given by $[A_A \cdot \lambda_4^{N'} - A_B \lambda_4^{D'}]$ where $\lambda_4^{N'}$ and $\lambda_4^{D'}$ are defined below. As $\lambda_4^{D'}$ is positive, the result proves that the sign of the welfare change has the same sign as $A_B/A_A - \lambda_4$, where $\lambda_4' = \lambda_4^{N'}/\lambda_4^{D'}$.

$$\lambda_4^{N'} = 4\left[-2v_BP_A'^2 \left[-24v_A^3 + 530v_A^2v_B + 132v_Av_B^2 + 43v_B^3\right] \cdot D'^4 + 4v_B^2P_A'^3\left[49v_A + 13v_B\right] \cdot D'^3 + P_A' \left[2v_A^5 - 183v_A^4v_B + 1946v_A^3v_B^2 + 309v_A^2v_B^3 + 262v_Av_B^4 - 52v_B^5\right] \cdot D'^5 - 2 \left[2v_A^6 - 89v_A^5v_B + 613v_A^4v_B^2 + 64v_A^3v_B^3 + 110v_A^2v_B^4 - 41v_Av_B^5 + 3v_B^6\right] \cdot D'^6\right]P_B'$$

$$+ 2\left[2P_A'^2 \left[-8v_A^3 + 381v_A^2v_B + 146v_Av_B^2 + 62v_B^3\right] \cdot D'^3 + P_A' \left[64v_A^4 - 1448v_A^3v_B - 700v_A^2v_B^2 - 397v_Av_B^3 + 43v_B^4\right] \cdot D'^4 - v_BP_A'^3\left[133v_A + 25v_B\right] \cdot D'^2 + 2 \left[-32v_A^5 + 460v_A^4v_B + 226v_A^3v_B^2 + 158v_A^2v_B^3 - 39v_Av_B^4 + v_B^5\right] \cdot D'^5\right]P_B'^2$$

$$+ \left[-2P_A' \left[60v_A^3 + 72v_A^2v_B + 29v_B^2\right] \cdot D'^2 + 3P_A'^2\left[10v_A + v_B\right] \cdot D' + 112v_A^3 \cdot D'^3 + 224v_B^3 \cdot v_BD'^3 + 90v_A^2v_B^2 \cdot D'^3 - 9v_B^3 \cdot D'^3\right]P_B'^3 \left[P_A' - 2v_AD\right] + 4 \left[4v_A + v_B\right] \cdot D' \left[P_A' - 2v_AD\right] \cdot D'^2 \cdot P_B'^4$$

$$+ 8\left[2v_BP_A'^2 \left[-18v_A^3 + 242v_A^2v_B + 43v_Av_B^2 + 3v_B^3\right] \cdot D'^4 - 32v_B^2P_A'^3\left[3v_A + v_B\right] \cdot D'^3 + P_A' \left[-3v_A^5 + 129v_A^4v_B - 859v_A^3v_B^2 + 15v_A^2v_B^3 - 38v_Av_B^4 - 12v_B^5\right] \cdot D'^5 + 2 \left[3v_A^6 - 60v_A^5v_B + 260v_A^4v_B^2 - 62v_A^3v_B^3 + 27v_A^2v_B^4 - 10v_Av_B^5 + v_B^6\right] \cdot D'^6\right]v_BD' \quad (C.21)
\begin{align*}
\lambda_4^D &= 4 \left[ 2v_B P_A^3 \left[ 12v_A^2 + 210v_A v_B + 43v_B^2 \right] D^3 - 52v_B^2 P_A^4 D^4 \\
&\quad + P_A^2 \left[ 7v_A^4 - 124v_A^3 v_B - 1269v_A^2 v_B^2 - 422v_A v_B^3 + 52v_B^4 \right] D^6 \\
&\quad + 2P_A' \left[ -14v_A^5 + 105v_A^4 v_B + 837v_A^3 v_B^2 + 324v_A^2 v_B^3 - 91v_A v_B^4 + 3v_B^5 \right] D^5 \\
&\quad + 4v_A \left[ 7v_A^5 - 30v_A^4 v_B - 203v_A^3 v_B^2 - 77v_A^2 v_B^3 + 38v_A v_B^4 - 3v_B^5 \right] D^5 \right] P_B' \\
&\quad - 2 \left[ P_A^2 \left[ 11v_A^3 + 157v_A^2 v_B + 124v_B^2 \right] D^2 - P_A' \left[ 44v_A^3 v_B + 354v_A^2 v_B^2 + 454v_A v_B^3 - 43v_B^4 \right] D^3 \\
&\quad - 25v_B D' P_A^3 + [40v_A + 256v_A v_B + 393v_A^2 v_B^2 - 83v_A v_B^3 + 2v_B^4] D^4 \right] P_B'' \left[ P_A' - 2v_A D' \right] \\
&\quad - \left[ 4P_B^2 \left[ P_A - 2v_A D \right]^3 \right] \\
&\quad + 8 \left[ 32v_B P_A^3 D^3 - 2v_B^2 P_A^3 \left[ 6v_A^2 + 124v_A v_B + 3v_B^2 \right] D^4 \\
&\quad - v_B P_A^2 \left[ 9v_A^4 - 48v_A^3 v_B - 705v_A^2 v_B^2 + 8v_A v_B^3 + 12v_B^4 \right] D^6 \\
&\quad + P_A' \left[ v_A^6 - 31v_A^5 v_B + 67v_A^4 v_B^2 + 881v_A^3 v_B^3 - 86v_A^2 v_B^4 - 26v_A v_B^5 + 2v_B^6 \right] D^6 \\
&\quad + 2v_A [v_A^6 - 14v_A^5 v_B + 17v_A^4 v_B^2 + 206v_A^3 v_B^3 - 47v_A^2 v_B^4 - 4v_A v_B^5 + v_B^6] D^7 \right] \tag{C.22}
\end{align*}